

375. A cornfield has 1000 cornstalks. When the farmer stands at a cornstalk at the corner of the field he notices that some of the cornstalks line up with the one he is standing at. On closer examination it turns out that the number of these lines which contain an odd number (such as 1, 3, 5, ...) of other cornstalks is odd. Is this true no matter which cornstalk he stands at? Why or why not?

376. Given n sacks each holding the same number of apples. On the first day an apple is removed from one sack. On the second day an apple is removed from each of 2 sacks; and so on until the n th day when one apple is removed from each of the n sacks. The sacks are now all empty. For which n is this possible, and how is it to be done?

377. Let x_i, y_i ($i = 1, 2, \dots, n$) be real numbers such that $x_1 \geq x_2 \geq \dots \geq x_n$ and $y_1 \geq y_2 \geq \dots \geq y_n$.

Prove that, if $z_1, z_2, \dots, z_n$ is any rearrangement of $y_1, y_2, \dots, y_n$ then

$$\sum_{i=1}^n (x_i - y_i)^2 \leq \sum_{i=1}^n (x_i - z_i)^2$$

378. Let $a_1, a_2, a_3, \dots$ be any infinite sequence of strictly positive integers such that $a_k < a_{k+1}$ for all k . Prove that infinitely many a_m can be written in the form $a_m = xa_p + ya_q$ with x, y strictly positive integers and $p \neq q$.

379. On the sides of an arbitrary triangle ABC , triangles ABR , BCP and CAQ are constructed externally with $\angle PBC = \angle CAQ = 45^\circ$; $\angle BCP = \angle QCA = 30^\circ$; $\angle ABR = \angle RAB = 15^\circ$.

Prove that $\angle QRP = 90^\circ$ and $QR = RP$.

380. When 4444^{4444} is written in decimal notation, the sum of its digits is A . Let B be the sum of the digits of A . Find the sum of the digits of B .

Solutions to Problems from Vol. 13 No. 2

345. During a trial, three different witnesses A, B and C were called one after the other, and asked the same questions. In each case, each witness answered "yes" or "no", and the following facts were noted:

- All questions answered "yes" by both B and C were also answered "yes" by A ;
- every question answered "yes" by A was also answered "yes" by B ;
- every question answered "yes" by B was also answered "yes" by at least one of A and C .

Show that the witnesses A and B agreed in their answers to all questions.

Solution: D. Dowe (Geelong Grammar) writes:-

Using clues (a) and (c), we find if B says "yes", so must A. From (c), if B says "yes", so does C or A; and if C does, we find from (a) that A says "yes" also. Therefore, if B says "yes", so does A.

Using clue (b), if A says "yes", so does B. Thus, we can see that A and B both said "yes" to the same questions, and therefore agreed in all their answers.

Also solved by P. Stott (Newington) and one other who omitted to write his (or her) name on the solutions supplied. I shall refer to this solver as "Q" below.

346. Five 2-digit numbers are made up from the digits 0, 1, . . . , 9, each digit being used exactly once. What is the largest possible value of the product of the five numbers?

Solution: P. Rider (St Leo's College, Wahroonga) writes:-

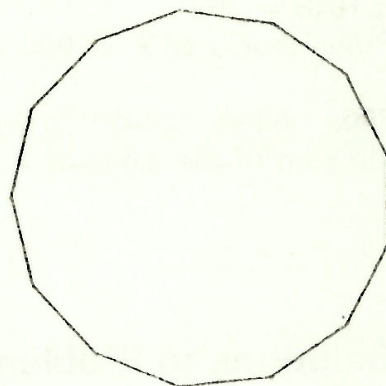
We require the products of the ten's digits to be as large as possible. Therefore the tens digits will be (9, 8, 7, 6, 5). Therefore the units digits will be (4, 3, 2, 1, 0). The largest of the units digits must be placed with the smallest of the tens digits. So that the largest of the tens could be multiplied by the largest of the units digits, as the tens digits ascend in value the units digits will descend in value. The numbers will therefore be (90, 81, 72, 63, 54).

Therefore the largest possible value of the products of the five numbers is $90 \times 81 \times 72 \times 63 \times 54 = 1,785,641,760$.

This is convincing enough (even if strictly not a complete proof) and in my judgement the sort of answer which the question warrants.

Also solved by J. Taylor (Woy Woy High), D. Dowe, P. Stott and Q.

347. Is it possible to select four vertices of the following regular 13-sided polygon so that the four sides and two diagonals of the quadrilateral formed by the four chosen vertices have different lengths?



Solution:

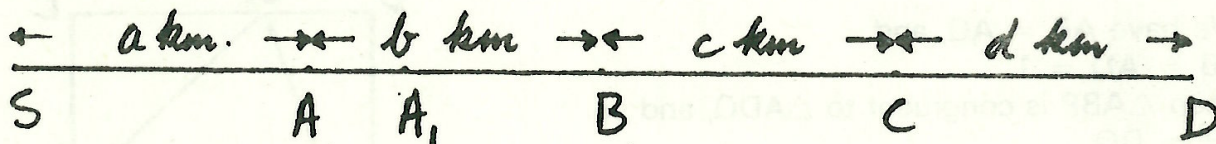
If the vertices are labelled in order 1, 2, 3, . . . , 13 round the figure, the required conditions are satisfied by choosing vertices 1, 2, 4, 10; or, alternatively, by choosing vertices 1, 2, 5, 7.

Solved by D. Dowe, P. Stott.

348. Four explorers are going to make a trip into the desert. Each man can carry enough water to last ten days. Each man can walk 24 kilometres a day. Obviously if all four stay together they can manage a trip of only five days into the desert, leaving enough water to return. If our explorers are thinkers, how far can they manage to get into the desert before they have to return, assuming that the desert is so uninhabited that it is safe to leave water behind for the return trip, but no explorer can return to civilisation to replenish his supply and then return to the desert.

Solution: What follows is essentially the solution of D. Dowe, to which I have taken the liberty of adding additional explanation.

Each man starts with 240 units of water where one unit of water is the amount used by one man in walking one kilometre.



All four go to A, one returns. The remaining three go to B, one returns. The two proceed to C, one returns. The last man goes to D and returns. Thus SA is traversed 8 times (twice by each of the four men), AB 6 times, BC 4 times and CD twice. At A, 3a units of water must be dumped for the return trip AS for each of the three men who travel past A. Similarly 2b units are stored at B, and c units at C.

I now claim that the value of "a" to maximise the distance SD is $240/8 = 30$. The 240 units of water taken by the first man are exactly used up by: 120 units used by the four men on the outward trip SA, 90 units dumped at A, and 30 units consumed by the first man's return trip. This means that the remaining 3 men still have a full load of water as they proceed beyond B. If "a" were less than 30, the additional water saved could therefore not be carried by the three proceeding further. On the other hand, if the first man went to A_1 , where $SA_1 \geq 30$ km, the three others would have the amount of water available for their trip beyond A reduced by the first man's requirements for AA_1 and return, A_1A . Hence their possible range must be reduced. Similar reasoning shows that b must be $240/6 = 40$, $c = 240/4 = 60$ and $d = 240/2 = 120$ to achieve maximum distance. Thus the team can penetrate $120(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4})$ km into the desert, a distance of 250 km. As David points out, with n men the maximum distance is $120(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + 1/n)$ which can be made as large as desired by taking n sufficiently large.

Also solved by P. Stott.

349. Solve the system of equations

$$x^{\log y} + y^{\log x} = 110, \quad xy = 1000.$$

Solution: P. Stott writes:-

To simplify matters, set $\log x = X$, $\log y = Y$.

So $X + Y = \log 1000 = 3$, and $Y = 3 - X$.

Now, $x = 10^X$ and $y = 10^Y = 10^{3-X}$, so

$$10^{X(3-X)} + 10^{\frac{1}{2}X(3-X)} = 110.$$

Put $10^{\frac{1}{2}X(3-X)} = n$. Then $n^2 + n - 110 = 0$, so $n = 10$ or -11 . However, $n = -11$ is impossible, since a power of 10 must be positive. Thus $\frac{1}{2}X(3-X) = 1$, giving $X = 1$ or 2 . So, the solution of the system of equations is $x = 100$, $y = 10$, or $x = 10$, $y = 100$.

Also solved by M. Dyer, D. Dowe, H. Nayna (Parramatta Marist High).

350. Let ABCD be a square of side 1 cm. Suppose P lies on BC, Q lies on DC, and that $AP = AQ$. Show that the perimeter of the triangle APQ is not more than $2 + \sqrt{2}$ cm.

Solution: M. Dyer writes:-

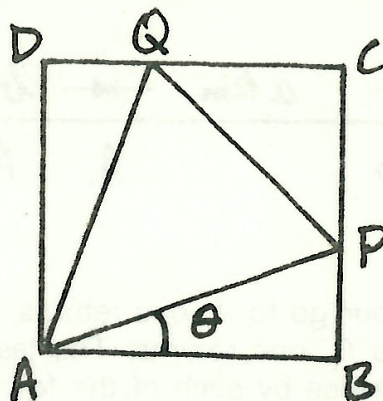
We have $AP = AQ$, and

$AB = AD = 1$.

Then $\triangle ABP$ is congruent to $\triangle ADQ$, and so

$BP = DQ$.

Hence $QC = 1 - DQ = 1 - BP = CP$.



Now if we let $\angle BAP = \theta$, then we have $BP = \tan \theta$, so $CP = 1 - \tan \theta$, and $QP = \sqrt{2}(1 - \tan \theta)$.

Also, $PA = QA = \sec \theta$.

Then the perimeter P of the triangle APQ is

$$P = P(\theta) = AP + AQ + QP = 2 \sec \theta + \sqrt{2}(1 - \tan \theta),$$

$$\begin{aligned} \text{so } dp/d\theta &= 2 \sec \theta \tan \theta - \sqrt{2} \sec^2 \theta \\ &= \sqrt{2} \sec \theta (\sqrt{2} \tan \theta - \sec \theta) \\ &= \sqrt{2} \sec^2 \theta (\sqrt{2} \sin \theta - 1) \\ &\leq 0 \quad \text{since } 0 \leq \theta \leq 45^\circ. \end{aligned}$$

$\therefore$ The maximum value of $P(\theta)$ is $0 \leq \theta \leq 45^\circ$ occurs at $\theta = 0$. This value is $P(0) = 2 + \sqrt{2}$ as was to be shown.

D. Dowe also supplied a solution using calculus. A solution not using any calculus was given by P. Stott.

351. A product $x \circ y$ is defined for all pairs of real numbers x, y so that the following hold for any x, y, z :

- i) $x \circ y = y \circ x$
- ii) $(x \circ y) \circ z = x \circ (y \circ z)$
- iii) $(x \circ y) + z = (x + z) \circ (y + z)$

What is the value of $99 \circ 100$?

Solution: P. Stott writes:-

Consider the quantity $(x \circ -x)$.

By rule (ii) $-1(x \circ -x) = (-x \circ x)$, which by rule (i) is equal to $(x \circ -x)$. It follows that $(x \circ -x) = 0$. Now, according to rule (iii), $(\frac{1}{2} \circ -\frac{1}{2}) + 99\frac{1}{2} = (99\frac{1}{2} + \frac{1}{2}) \circ (99\frac{1}{2} - \frac{1}{2})$. Since $(\frac{1}{2} \circ -\frac{1}{2}) = 0$ it follows that $99 \circ 100 = 99\frac{1}{2}$. (Indeed, it can be seen that $x \circ y = \frac{1}{2}(x + y)$, but that wasn't asked for.)

M. Dyer points out that the result cannot be obtained if (ii) is assumed only for $z \geq 0$ since the definition

$$x \circ y = \min(x + a, y + a)$$

would then satisfy all conditions, and by choosing the number a appropriately in this definition one can make $99 \circ 100$ equal to any desired real number.

Also solved by D. Dowe, H. Nayna.

352. How many square numbers are there whose digits, when written in base 10 notation, are three hundred 1's, and some number of 0's?

Solution: D. Dowe writes:-

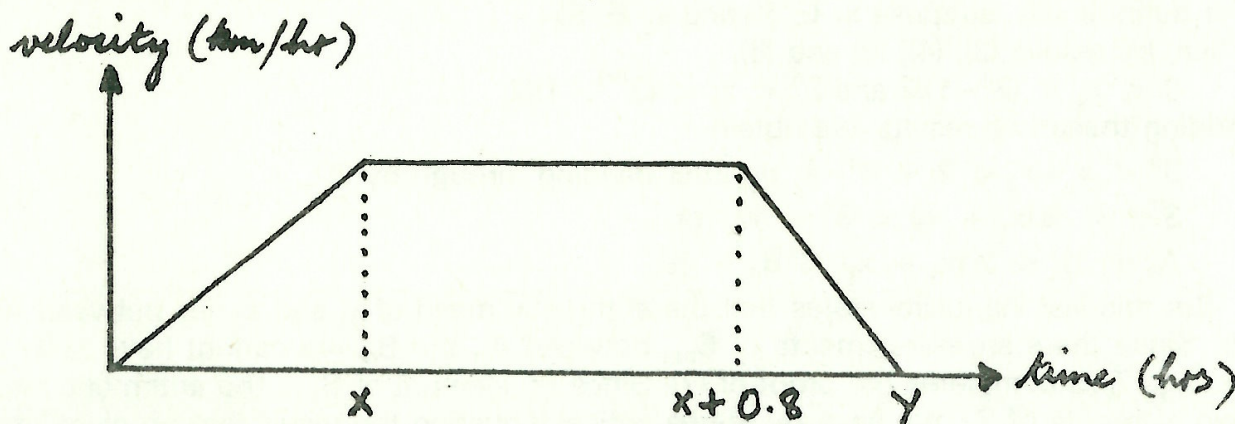
There are no such numbers. This is so because the sum of the number's digits is 300, which is a multiple of 3 but not a multiple of 9. This implies that any such number is divisible by 3, but not divisible by 9. No such number can be a square.

Also solved by M. Dyer, P. Stott, H. Nayna.

353. In travelling from A to B, a distance of 100 km, a train accelerates uniformly, travels 80 km at a constant speed of 100 km/h, and then decelerates uniformly. How long does the trip take?

Solution: D. Dowe writes:-

The given data enables us to construct this graph:



The distance travelled equals the area under the graph.

$$\text{Thus, } 100 = (100x)/2 + 0.8 \times 100 + [100(y - 0.8 - x)]/2$$

$$\therefore 100 = 50x + 80 + 50y - 40 - 50x$$

$$\therefore 100 = 40 + 50y$$

$$\text{giving } 60 = 50y$$

$$\text{and } y = 1.2 \text{ hrs.}$$

Thus, the trip takes 1 hr 12 mins.

Also solved by M. Dyer, P. Stott.

354. Prove that it is possible to select 2^k different numbers $a_1, \dots, a_{2^k}$ from the set $\{0, 1, 2, \dots, 3^k - 1\}$ in such a way that none of the a 's is the arithmetic mean of any other two.

Solution: M. Dyer writes:-

We will show that the required result is true by constructing such a set.
Let $S = \{0,1\}$, and define

$$S'_n = \{x: (x - 3^n) \in S_{n-1}\} \quad \dots (1)$$

$$\text{and } S_{n+1} = S_n \cup S'_n \quad \dots (2)$$

(1) and (2), used in conjunction with the definition of S_1 , can be used to generate a succession of sets $S_1, S_2 \dots S_n$.

For example, $S'_1 = \{3,4\}$ so $S_2 = \{0,1,3,4\}$.

Then $S'_2 = \{9,10,12,13\}$ so $S_3 = \{0,1,3,4,9,10,12,13\}$ etc.

The smallest element of S_n is 0.

(3)

The largest element of S_n is $A_n = 1 + 3 + \dots + 3^{n-1} = (3^n - 1)/2$

(4)

The smallest element of S'_n is $B_n = 3^n$, and

(5)

The largest element of S'_n is $C_n = (3^{n+1} - 1)/2$.

(6)

These results, if not immediately apparent, can be easily proved by induction.

Clearly, we have $0 < A_n < B_n < C_n$. Also, since every member of S_n is an integer $< A_n + \frac{1}{2} < 3^n - 1$ we also have $S_n \subset R_n$, where $R_n = \{0,1,2 \dots 3^n - 1\}$.

We will now show that if no element of S_n is the arithmetic mean of any other two, the same is true of S_{n+1} .

(7)

Let x_1 and x_2 be two elements of S_{n+1} . The result is obviously true if $x_1 \in S_n$ and $x_2 \in S_n$, or if $x_1 \in S'_n$ and $x_2 \in S'_n$. For the elements of S'_n are merely those of S_n increased by 3^n , and the result has been assumed to hold for S_n .

The only other case to be considered is if exactly one of x_1 and x_2 is an element of S_n . For definiteness, suppose $x_1 \in S_n$ and $x_2 \in S'_n$.

Then, by results (3), (4), (5) and (6),

$$0 < x_1 < (3^n - 1)/2 \text{ and } 3^n < x_2 < (3^{n+1} - 1)/2.$$

Adding these two results, we obtain

$$3^n < x_1 + x_2 < 2 + 3^n - 1, \text{ or, after dividing through by 2,}$$

$$3^n/2 < \frac{1}{2}(x_1 + x_2) < 3^n - \frac{1}{2}, \text{ or}$$

$$A_n + \frac{1}{2} < \frac{1}{2}(x_1 + x_2) < B_n - \frac{1}{2}.$$

But this last inequality states that the arithmetic mean of x_1 and x_2 lies between A_n and B_n . Since there are no elements of S_{n+1} between A_n and B_n , we cannot have $\frac{1}{2}(x_1 + x_2) \in S_{n+1}$. This completes the proof of (7). Since no element of S_1 is the arithmetic mean of two elements of S_1 , we have by mathematical induction the result that no element of S_n is the arithmetic mean of any other two.

To prove the original result, it remains to show that S_n has 2^n members. But this follows at once, since S_1 has 2 members, and S_{n+1} has twice as many elements as S_n , since S_n and S'_n have the same cardinal number, and $S_{n+1} = S_n \cup S'_n$.

Also solved by D. Dowe, P. Stott and H. Nayna who exhibited the same set of numbers S_n . It is perhaps worth pointing out that these numbers are all the natural numbers $< 3^n$ whose expressions using base three instead of ten, do not use the digit 2. This can be used to rewrite the proof.

355. Let p be a prime number, and let $\binom{n}{p} = \frac{n(n-1) \dots (n-p+1)}{1 \cdot 2 \dots p}$, $[n/p]$ the largest integer not greater than n/p . Prove that $\binom{n}{p} - [n/p]$ is divisible by p .

Solution: M. Dyer writes:

Assume n is a non-negative integer, and write

$$n = kp + r, \quad 0 \leq r < p \quad \dots (1)$$

where k and r are positive integers determined uniquely by (1).

Of the numbers $n, (n-1), \dots, (n-p+1)$, exactly one, namely kp , will be congruent to $0 \pmod{p}$, and the remaining $p-1$ numbers will be congruent mod p in some order to $1, 2, \dots, (p-1)$.

$$\therefore n(n-1) \dots (n-p+1)/p \equiv [1 \cdot 2 \dots (p-1)]k \pmod{p} \quad \dots (2)$$

From (1), we obtain

$$n/p = k + r/p, \quad 0 \leq r/p < 1. \quad \text{i.e. } [n/p] = k \quad \dots (3)$$

Using (2) and (3), we find

$$n(n-1) \dots (n-p+1)/p - 1 \cdot 2 \cdot 3 \dots (p-1) [n/p] \equiv 0 \pmod{p}$$

i.e. $[1 \cdot 2 \dots (p-1)][\binom{n}{p} - [n/p]]$ is divisible by p , where the second factor is integral since $\binom{n}{p}$ and $[n/p]$ are both integers. Since p is prime, $1 \cdot 2 \dots (p-1)$ is relatively prime to p and $\binom{n}{p} - [n/p]$ must be divisible by p . This is the required result.

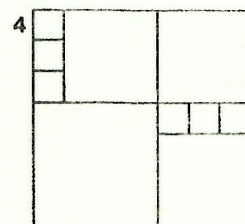
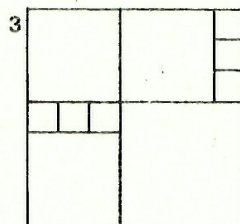
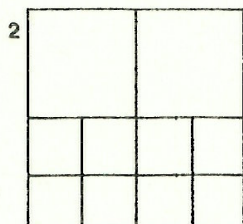
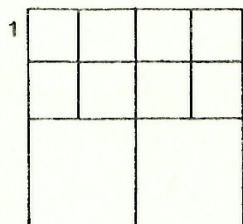
Also solved by D. Dowe, P. Stott, H. Nayna.

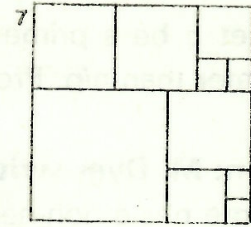
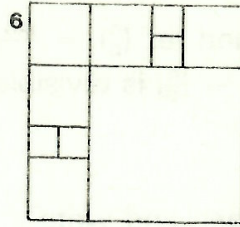
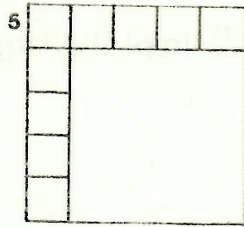
356. A suitor asking for the hand of the king's daughter is given the following task:

Divide the square wall of the princess's room into ten smaller squares, a different way on each day for a week. No square should have the same 4 vertices as any square used on previous days. Is it possible for the suitor to marry the princess or will he end up on the chopping block?

Solution: D. Dowe writes:-

I have found a satisfactory solution, so I think the prospect of wedlock is inevitable.





Also solved by M. Dyer, H. Nayna.

Solvers of earlier problems.

The following contributions were received too late for acknowledgment in the last issue:-

From M. Dyer and H. Nayna, solutions of problems 337 through 344 (all excellent), D. Dowe, Problem 341.