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MATHEMATICAL MODELLING

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In trying to deal with questions such as:

1. In a new housing sub-division, how many letter boxes should Australia Post provide and where should they be placed?

2. What is the most efficient way to divide the fuel between the stages of a multistage rocket?

3. Is it a good idea to use a dialysis machine to help remove a drug when we have somebody suffering from a drug overdose?

or many thousands of other similar questions, you will probably use mathematical modelling. Mathematical modelling is an art. The artist is someone who tries to mimic a real-life situation using the language of mathematics with the hope that something useful will come out of the investigation.

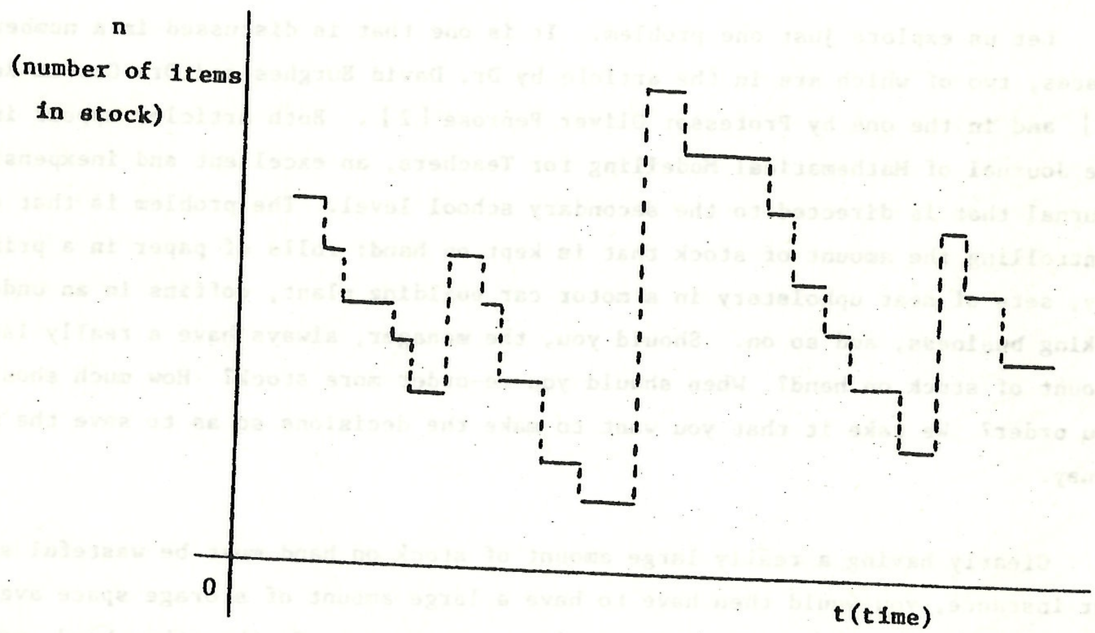
It often occurs that, if the mimicry is too exact, then the resulting mathematical problem is very difficult to solve while, if the mimicry is of a too rough and ready a sort, then the results are not very useful in dealing with the real-life problem. Herein lies some of the art of the mathematical modeller: in deciding which aspects of the problem should be concentrated upon and which ones should be ignored. A way of proceeding that has been found to be very fruitful is to set up and investigate, first of all, a simple mathematical model, one which includes just the major features of the problem and, probably, expresses those in a simplified way. Even this simple model will sometimes shed some light on the problem or, perhaps, will indicate by its inadequacy that something that was previously thought to be unimportant must be included. After the simple model, an improved model can be considered and then, if necessary, one still further refined, and so on, until we have a satisfactory resolution of the original problem.

In a short article it is not possible to discuss and illustrate all of the things that go into doing mathematical modelling as this covers so much. Indeed, if we wanted to make a gross exaggeration, we could say "all of applied mathematics is mathematical modelling". Be that as it may, we can note, as an indication of the importance with which the acquisition of mathematical modelling skills is held, that this year at the University of New South Wales we have introduced for first year students a new optional mathematics subject and one third of it is devoted to an introduction to mathematical modelling. (Of course, students also do the usual first year mathematics subject.) Even this introduction has not time to touch on many matters that can be of importance in mathematical modelling.

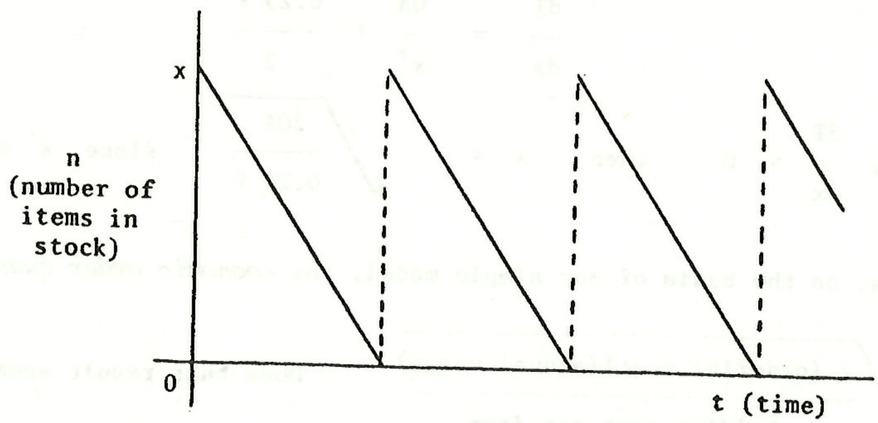
Let us explore just one problem. It is one that is discussed in a number of places, two of which are in the article by Dr. David Burghes and Dr. Graham Read [1] and in the one by Professor Oliver Penrose [2]. Both articles appear in the Journal of Mathematical Modelling for Teachers, an excellent and inexpensive journal that is directed to the secondary school level. The problem is that of controlling the amount of stock that is kept on hand: rolls of paper in a printery, sets of seat upholstery in a motor car building plant, coffins in an undertaking business, and so on. Should you, the manager, always have a really large amount of stock on hand? When should you re-order more stock? How much should you order? We take it that you want to make the decisions so as to save the most money.

Clearly having a really large amount of stock on hand must be wasteful since, for instance, you would then have to have a large amount of storage space available and this might involve buying an extra warehouse. On the other hand, you do not want to have so little stock on hand that you are always being worried about the stock's running out and so disrupting your business. There is a cost, called the *holding cost*, associated with having stock on hand: there is still some cost for storage and there is the cost of the money spent to buy the stock. This latter cost is because either you have borrowed that money, in which case you will be paying interest, or else you have had that money on hand and you are losing out on the interest you could have earned by lending it to someone else. It is quite a complicated thing to decide what the holding cost is; let us make a simplifying assumption that the annual holding cost of an item in the store is 25% of the cost price of the item. Suppose each item costs \$P. The other major cost is called the *ordering cost* and is composed of the cost of all the paperwork associated with placing and receiving the order and the cost of delivery. It could be that, within fairly wide limits, the cost of delivery is the same: one semi-trailer is going to be needed for a small or a large order. At any rate, let us make another simplifying assumption that the ordering cost is constant, \$Q, no matter what the size of the order.

If you did not try to be systematic and merely re-ordered at whim, then a graph of the number, n, of items in stock against time could look something like this:



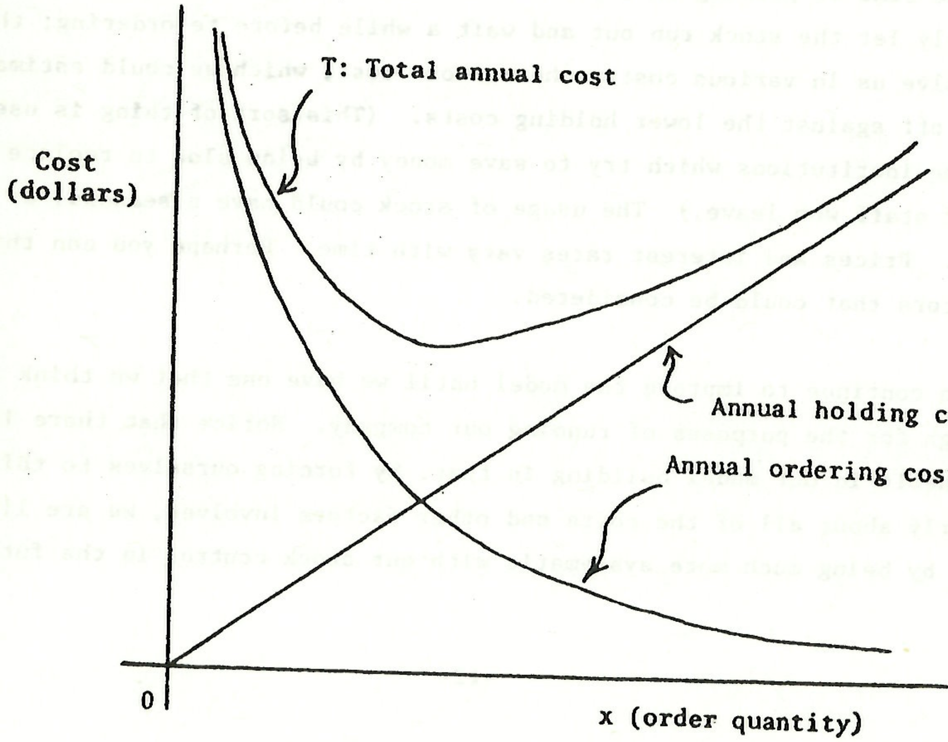
(The dashed lines are not parts of the graph but are put in to help the eye.) Each vertical jump upwards indicates a re-stocking while those downwards show usage of the stock. It would be difficult to mimic this closely and remember that we would like to be systematic. In this first model of the problem, let us make the following simplifying assumptions. First assume that we always order the same amount and that we place the orders regularly. Secondly, to save us from holding too much stock, assume that each order is placed just as the stock runs out and that the delivery is made instantaneously. Thirdly, assume that we use the stock at a constant rate and that we can think of the number of items in stock as being a continuous variable. With these assumptions, we have an idealized graph of the form:



If we know that we use A items each year then, if each order is for x items, there will be (A/x) orders each year. Thus the annual ordering cost will be \$(QA/x). From the graph it is clear that the average number of items of stock held through the year is x/2, since the graph is always dropping steadily from x to zero between restockings. Thus the annual holding cost will be \$(0.25)Px/2. If the total annual cost is \$T, we have

$$T = \frac{QA}{x} + \frac{0.25\ P\ x}{2}\ .$$

A graph of T against x will have the general form as shown below and it indicates that T will have a minimum for some positive value of x.



We can find this minimum by using the differential calculus.

$$\frac{dT}{dx} = -\frac{QA}{x^2} + \frac{0.25\ P}{2}$$

Thus $\frac{dT}{dx} = 0$ when $x = \sqrt{\frac{2QA}{0.25\ P}}$, since x is positive.

Thus, on the basis of our simple model, the *economic order quantity* is

$$\sqrt{\frac{2\ (\text{ordering cost})(\text{annual usage})}{\text{holding cost per item}}}$$

. Does this result seem reasonable?

From it we see that a higher price for each item means a smaller order quantity. A higher annual usage means a larger order quantity but it would take a fourfold increase in usage before the size of the order would be doubled. Both of these predictions from the model seem to be the sort of thing that we would expect. Notice that, by working generally, we have a result that we can apply to any stock control problem where our assumptions do not seem to be too far away from reality.

We have investigated a simple model. For an improved model, we could take into account various things which would make it mimic the real-life situation more closely. There is a *lead time*, which might depend on various factors, between the time of placing an order and the time the goods arrive. We could deliberately let the stock run out and wait a while before re-ordering; this would involve us in various costs, the run-out cost, which we could estimate and trade off against the lower holding costs. (This sort of thing is used a lot by some institutions which try to save money by being slow to replace members of staff who leave.) The usage of stock could have a seasonal or other variation. Prices and interest rates vary with time. Perhaps you can think of other factors that could be considered.

We can continue to improve the model until we have one that we think is good enough for the purposes of running our company. Notice that there is a hidden benefit in our model building in that, by forcing ourselves to think more clearly about all of the costs and other factors involved, we are likely to end up by being much more systematic with our stock control in the future.

We are also in a better position to consider, for instance, whether we disagree or not with the idea of always having a bit of extra stock on hand "just in case".

Mathematical modelling can be simpler or more complicated than in the example above. It is always interesting, however, since it is always concerned with trying to do something useful.

References

[1] D.N. Burghes and G.A. Read, "Mathematical modelling: Editorial statement", *Journal of Mathematical Modelling for Teachers*, Vol. 1, No. 1 (June, 1978), pages 1 - 10.

[2] O. Penrose, "How can we teach mathematical modelling?", *Journal of Mathematical Modelling for Teachers*, Vol. 1, No. 2 (December, 1978), pages 31-42.

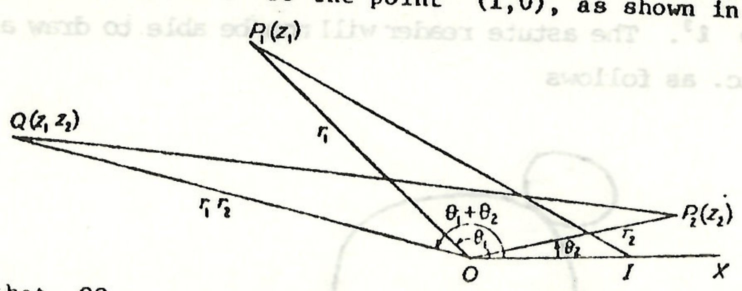
MORE FROM TREVOR'S H.S.C. CORNER

Geometrical Representation of Multiplication of Complex Numbers.

Given $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then by de Moivre's Theorem.

$$z_1z_2 = r_1r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)).$$

Let z_1, z_2 and z_1z_2 be represented by the points P_1, P_2 and Q on an Argand diagram. Also let I be the point $(1,0)$, as shown in the diagram below



Observe that $OQ = r_1r_2 = OP_1 . OP_2$, we have $OQ:OP_2 = OP_1:OI$.

Also the angle, measured in the positive sense, from OP_2 to OQ is equal to the angle from OI (or OX) to OP_1. The triangles OP_2Q and OIP_1 are therefore similar.

Thus when P_1 and P_2 are given, the point Q, representing z_1z_2 may be constructed by drawing the triangle OP_2Q similar to the triangle OIP_1.