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QUANTIFIERS, CHOICE AND GAMES

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This note is a sequel to the article on underpromotions in chess published in the previous issue of *Parabola*. The logic behind statements like "White to play and win" and "With White to play, this position is drawn" is discussed here.

Six logical operators occur in this topic. These are:

$\wedge$	and	$\vee$	or
$\neg$	not	$\leftrightarrow$	if and only if
$\exists$	there exists	$\forall$	for all.

The last two are called quantifiers, and make sense in a context where some universe is specified either implicitly or explicitly. An elementary example is the statement (valid where the universe is the set of real numbers):

$$\forall x \exists y [y^3 = x].$$

This is not equivalent to the false statement:

$$\exists y \forall x [y^3 = x].$$

These symbols are not independent. For example:

$$A \vee B \leftrightarrow \neg(\neg A \wedge \neg B);$$

$$\forall x \Phi(x) \leftrightarrow \neg \exists x \neg \Phi(x).$$

Here A and B are any propostions and $\Phi(x)$ is any formula involving the variable x. Both *formula* and *variable* are given somewhat technical meanings in the theory of logic. Thus the former equivalence is that between "A or B" and "it is false that both A is false and B is false", where A and B represent propositions whose internal structure is arbitrary. A formula already examined is $[y^3 = x]$. It involves the two variables x and y which are said to be *bound* by the quantifiers $\forall$ and $\exists$ respectively. In seeking greater generality, we allow abstract formulas $\Phi(x)$ or $\Psi(x, y)$, etc., in differing numbers of variables. An example with nine variables occurs below.

First consider the game of Noughts and Crosses, where for convenience the Nought player is required to make the first choice of a square. This choice is made out of a 3×3 grid of squares. The players alternatively choose and occupy squares. The first one to occupy three squares in a row wins; if neither player achieves this the game is drawn. There are 9 choices for the Nought player's first move (here called n_1), 8 choices for the Cross player's response (c_1), 7 choices for the Nought player's second move (n_2), and so on. As some of these sequences of choices terminate prematurely in a win for one of the players, there are somewhat fewer than 362,880 valid games. We introduce the symbolism:

$N(n_1, c_1, n_2, c_2, n_3)$ means that Nought wins after the sequence n_1, c_1, n_2, c_2, n_3 of choices;

$C(n_1, c_1, n_2, c_2, n_3, c_3)$ means that Cross wins after the sequence $n_1, c_1, n_2, c_2, n_3, c_3$ of choices;

etc.

So the (false) statement that the Nought player can win from the start of play amounts to:

$$\exists n_1 \forall c_1 \exists n_2 \forall c_2 \exists n_3 [N(n_1, c_1, n_2, c_2, n_3) \vee$$

$$\forall c_3 \exists n_4 [N(n_1, c_1, n_2, c_2, n_3, c_3, n_4) \vee \forall c_4 \exists n_5 [N(n_1, c_1, n_2, c_2, n_3, c_3, n_4, c_4, n_5)]]]$$

There is an analogous (false) statement that the Cross player can win from the start of play. The true analysis of the initial position amounts to the denial of both these statements. A similar analysis may be made of a statement about the outcome "with best play" of the current position in an unfinished game.

Now consider the more complex game of chess. The rules include a prohibition on castling whenever the king or the relevant rook has moved. There is the *en passant* capture rule. They give the player to move the right to claim a draw if there exists a move that recreates a position that that has occurred twice previously. There is also a right to claim a draw after 100 consecutive moves (50 by each player) have occured without any captures or pawn moves. As there are at most $30 + 6 \times 8 \times 2 = 126$ such events in a game, a sane game of chess cannot have more than 6300 moves. So the analogous logic underlies the analysis of the current position of any unfinished game. It is easily calculated that the statement (true or false?) "White can win from the initial position with best play" needs over 10,000,000 symbols to set out.

There are conventions of the *nul hypothesis* type about the interpretation of the pre-history of positions presented as problems, such as those in the previous *Parabola*.

Alternatively, we could define a *configuration* in chess to be a legal position on the board together with information about:

(1) which side is to move;

(2) which kings and rooks have not yet moved;

(3) which pawn, if any, advanced two squares on the previous move;

(4) the number of moves since the last pawn move or capture.

In principle, we determine which configurations are checkmates and mark them as *wins*. We then determine which positions are such that the player to move cannot avoid producing a *win* position for the opponent and mark them as *losses*. This process may be continued to classify all configurations as wins, losses or, in default of these, *draws*.