SOLUTIONS OF PROBLEMS FROM VOL. 23, 1.

Q. 696. k is a whole number. There is a pile of N coins shared amongst n brigands as follows:

The first brigand takes 1 coin and $\frac{1}{k}$th of the remainder.

The second brigand takes 2 coins and $\frac{1}{k}$th of the remainder.

The third brigand takes 3 coins and $\frac{1}{k}$th of the remainder.

. . . .

The rth brigand takes r coins and $\frac{1}{k}$th of the remainder

. . . .

Finally the nth brigand takes the remaining n coins.

Find n and N in terms of k.

Solution: Let N_r be the number of coins left when the rth brigand is about to take his apportioned share. Then after he takes r coins and $\frac{1}{k} \times (N_r - r)$ the number remaining is $N_{r+1} = \frac{k-1}{k}(N_r - r)$.

Hence $N_r = \left(\frac{k}{k-1}\right) N_{r+1} + r. = t\,N_{r+1} + r$ where $t = 1 + \frac{1}{k-1}$

We work backwards from $N_n = n$ (given) to obtain

$$N_{n-1} = t\,n + (n-1)$$

$$N_{n-2} = t\,N_{n-1} + (n-2) = t^2 n + t(n-1) + (n-2)$$

$$N_{n-3} = t\,N_{n-2} + (n-3) = t^3 n + t^2(n-1) + t(n-2) + (n-3)$$

and (after $n-r$ repetitions)

$$N_r = t^{n-r} n + t^{n-r-1}(n-1) + \ldots\ldots + t(r+1) + r$$

for each $r = 1, 2, \ldots n$.

$$\therefore\ t\,N_r = t^{n-r+1} n + t^{n-r}(n-1) + \ldots + t^2(r+1) + tr$$

Subtracting, $(t-1)N_r = t^{n-r+1}n - [t^{n-r} + \ldots + t] - r$

$$= t^{n-r+1} n - t \frac{(t^{n-r}-1)}{t-1} - r$$

$$= t^{n-r+1} \left(n - \frac{1}{t-1}\right) + \frac{t}{t-1} - r$$

$$\therefore \quad N_r = k \left(\frac{k}{k-1}\right)^{n-r} (n - k + 1) + (k-r)(k-1)$$

In particular, $N = N_1 = k \left(\frac{k}{k-1}\right)^{n-1} (n - k + 1) + (k-1)^2$

If $(k-1)$ is a whole number greater than 1 this can be an integer only if $(k-1)^{n-1}$ is a factor of $n - (k-1)$. But for any whole number n

$$(k-1)^{n-1} \geq 2^{n-1} > n - (k-1)$$

Thus $(k-1)^{n-1}$ can be a factor of $n - (k-1)$ only if $n - (k-1) = 0$, $k = n + 1$. In this case $N = n^2$ and for each r $N_r = (n + 1 - r) \times n$ so that the rth bandit takes $N_r - N_{r+1} = n$ coins.

Thus all possibilities are given by

$$k = 2, \quad n \text{ arbitrary}, \quad N = 2^n(n-1) + 1$$

$$k > 2, \quad n = k-1, \quad N = (k-1)^2 .$$

<u>Q. 697</u>. I have a number x which is divisible by 9 but not by 10. It has the property that when a certain one of its digits is deleted the result is the whole number $y = \frac{x}{9}$ which is also a multiple of 9. Prove that y too has the property that when one of its digits is deleted it decreases 9 times. Find all possible values of x.

<u>Solution</u>: Since the sum of digits is a multiple of 9 both before and after deletion the digit erased must be either 0 or 9. Also it is not the final digit, whose deletion always causes reduction by a factor ≥ 10. Call it d, and suppose it is in the nth place in x, so that

$$x = a \times 10^{n+1} + d \times 10^n + b \quad \text{where a and b are whole numbers,}$$

$$0 < b < 10^n, \quad b \text{ not divisible by } 10, \ n \geq 1.$$

Then $y = a \times 10^n + b$, and $x = 9y = 9a \times 10^n + 9b$.

Equating the two expressions for x yields

$$a\ (10 - 9)\ 10^n = 8\ b - d \times 10^n$$

The RHS would be negative if $d = 9$. Hence the deleted digit must be 0 and this simplifies to

$$a \times 10^n = 8\ b. \qquad (1)$$

If $n > 3$ we must have b divisible by 5^n and also by 2^{n-3}, whence b would be a multiple of 10. Hence we must have $n = 1, 2,$ or 3.

If $n = 1$, $a \times 10 = 8\ b$. b odd, the only solution
is $a = 4$, $b = 5$, $x = 405$, $y = 45$.

If $n = 2$ $a \times 100 = 8\ b$, b odd, the only solution
is $a = 2$, $b = 25$. $x = 2025$, $y = 225$.

If $n = 3$ $a \times 1000 = 8\ b \longleftrightarrow b = a \times 125$ with $a = 1, 3, 5,$ or 7.
$x = 10125$, 30375, 50625, or 70875.
$y = 1125$, 3375, 5625, or 7875.

From (1) since $b < 10^n$ we must have $a < 8$. Also $y = a \times 10^n + b = 9\ b$.

Hence deletion of the first digit a from y yields $b = \frac{y}{9}$.

(Correct solution from F. Antonuccio, St Gregory's College, Campbelltown).

<u>**Q. 698**</u>. Find the smallest whole number n such that $n^5 + 5^n$ is an exact multiple of 13.

<u>**Solution**</u>: F. Antonuccio (St Gregory's College, Campbelltown) writes:- Let P(n) and Q(n) be the remainders resulting from dividing n^5 and 5^n by 13 respectively. A table of these values may be constructed, with values for large n obtained from the repeating sequences generated:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
P(n)	1	6	9	10	5	2	11	8	3	4	7	12	0	1	6	9	10	5
Q(n)	5	12	8	1	5	12	8	1	5	12	8	1	5	12	8	1	5	12

Now, for $n^5 + 5^n$ to be an exact multiple of 13,
then $[\ P(n) + Q(n)\ ] \equiv 0 \pmod{13}$

The smallest value for n where this holds is circled above (where $n = 12$)

i.e. The smallest whole number n such that $n^5 + 5^n$ is an exact multiple of 13 is 12

(Comment: All possible values of n can be found by extending the table and using the periodicity in both $P(n)$ and $Q(n)$. Results: $n = n_o + 52k$ where k is any whole number and $n_o = 12, 14, 21,$ or 31. Ed.).

Q. 699. Let $z = 0.1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 14 \ldots$ where all the whole numbers are written consecutively after the decimal point.

(a) Find the millionth digit after the decimal point.

(b) Prove that z is not a rational number.

Solution:

(a) The numbers 1 - 9 account for 9 digits

"	10 - 99	" "	2×90	digits.
"	100 - 999	" "	3×900	digits.
"	1000 - 9999	" "	4×9000	digits.
"	10000 - 99999	" "	5×90000	digits.

Hence by the time we have got to 99999 we have used up 488889 digits. The remaining 511111 digits will require us to write another 85185 six-digit numbers and the first digit of the next one (i.e. of 185185). Hence the millionth digit is a 1.

(b) The decimal representations of rational numbers either terminate or recur. It is obvious that z does not terminate. It is not difficult to see that it does not recur. Suppose it recurred with some "period", P. By examining the first block of P digits one could determine the longest block of consecutive zeros occurring anywhere in z. Let it contain n zeros say. But the number 10^{n+1} contains $n + 1$ consecutive zeros and its digits eventually appear as a block in z. Contradiction.

A correct answer to the first part was sent by F. Antonuccio.

Q. 700. The train line between X and Y is 666 kilometres. Every kilometre there is a post giving the distances from

X and Y:- 0|666, 1|665, 2|664, ..., 666|0.

Only one digit is used on the half way post:- 333|333. On how many of the posts are exactly two digits used?

Solution: The post 442|224 is such a post. There are 8 posts between x and y using just the digits 2 and 4:- 222, 224, 242, 244, 422, 424, 442, and 444 kilometres from X. Similarly there are 8 posts using only 1 and 5, and 8 using only 0 and 6:- 0|666, 6|660, 60|606, 66|600, 600|66, 606|60, 660|6, 666|0.
In addition, the posts 3|663, 33|663, 663|3, and 633|33 use only 2 digits.
This makes a total of 28 posts.

Q. 701. Find all sets of integers x, y, z such that

$$\frac{1}{x}+\frac{1}{y} = \frac{1}{z}.$$

Solution:

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{z} \longleftrightarrow (x+y)z = xy \qquad (*)$$

Let the greatest common divisor of x,y be d. Let $x = dX$, $y = dY$. Then X and Y have no common factor except 1.

(*) becomes $(X+Y)z = dXY$ ($)

If $X + Y$ had any factor (p say) in common with X, p would also be a factor of $(X + Y) - X = Y$. We must have $p = 1$. Similarly $X + Y$ and Y are "relatively prime". Since $X + Y$ is a factor of dXY, but has no factor in common with X or Y, d must be a multiple of $X + Y$.

Let $d = k(X + Y)$. Now ($) gives $z = kXY$.

Hence $x = dX = kX(X+Y)$; $y = dY = kY(X+Y)$, $z = kXY$ where k, X, Y are integers. Any choice of k, X, Y, $(X + Y \neq 0)$ will yield a satisfactory set x, y, z since

$$\frac{1}{kx(x+y)} + \frac{1}{ky(x+y)} = \frac{y+x}{kxy(x+y)} = \frac{1}{kxy}$$

e.g. if one chooses k, x, y = 3, 7, -11 one obtains $x = -84$, $y = +132$, $z = -231$.

Correct solution from F. Antonuccio.

Q. 702. Let x and y be positive integers with $y > 1$. Prove that it is impossible that

$$\underbrace{\sqrt{x+\sqrt{x+\sqrt{x+\dots+\sqrt{x}}}}}_{y \text{ square roots}}$$

is a whole number.

Solution: First let us note that $u = \sqrt{x + \sqrt{x}}$ cannot be a whole number. If it is, then $u^2 = x + \sqrt{x}$ is also, whence $\sqrt{x}$ is a whole number, x a perfect square.

Say $x = a^2$.

Then $u^2 = a^2 + a = a(a + 1)$. Since a and a + 1 clearly have no common factor their product is a perfect square only if each of a, a + 1 are perfect squares. But (apart from 0, 1, ruled out by the stipulation that x is positive, therefore not zero) consecutive integers are not both perfect squares.

Now $v = \sqrt{x + \sqrt{x + \sqrt{x \ldots \sqrt{x}}}}$ for arbitrary y

(y square roots)

cannot be a whole number, since if it is, so is

$$v^2 - x = \sqrt{x + \sqrt{x + \ldots \sqrt{x}}}$$

y - 1 square roots

and, repeating the process, so eventually is $\sqrt{x + \sqrt{x}}$.

But as we have seen this is never the case.

(Correct solution from F. Antonuccio).

Q.703. All three sides of a right angled triangle are whole numbers of centimetres, the area is A sq centimetres and the perimeter is P centimetres. If A = P find all possible side lengths.

Solution: Let x, y, z be the numbers of centimetres in the sides so that $x^2 + y^2 = z^2$. The formula giving all solutions in whole numbers is

$x = 2quv$, $y = q(u^2 - v^2)$, $z = q(u^2 + v^2)$ where q, u, v are any positive integers, u > v. (Also x,y may be interchanged, but this is of no importance here).

Since $A = \frac{1}{2} x y \text{ cm}^2$ and $P = (x + y + z)$ cm

$$A = P \longleftrightarrow q^2 u v (u^2 - v^2) = q(2u v + 2u^2)$$

$$\longleftrightarrow q v (u - v) = 2.$$

There are three solutions of this: q = 1, v = 2, u = 3; q = 1, v = 1, u = 3; or q = 2, v = 1, u = 2.

The first yields (x, y, z) = (12, 5, 13)

The second yields (x, y, z) = (6, 8, 10).

The third yields (x, y, z) = (8, 6, 10)

Hence there are only 2 essentially different triangles with the given property.

(Correct answer from F. Antonuccio.)

Q.704. Solve the equation

$$\sqrt{c - \sqrt{c + x}} = x.$$

Solution: We assume that c is a real number such that the equation has (a) real solution(s). Clearly $x \geq 0$ but $x^2 - c = -\sqrt{c + x}$ (1)
which is not positive.
Squaring (1) and collecting terms yields

$$c^2 - (2x^2 + 1)\, c + (x^4 - x) = 0$$

which can be solved for c in terms of x to yield

$$c = \frac{(2x^2 + 1) \pm \sqrt{(4x^4 + 4x^2 + 1) - 4(x^4 - x)}}{2} = \frac{(2x^2 + 1) \pm \sqrt{4x^2 + 4x + 1}}{2}$$

$$= (x^2 + x + 1) \quad \text{or} \quad x^2 - x .$$

Apart from the trivial case $c = 0$, $x = 0$, $c = x^2 - x$ can be rejected since it gives $x^2 - c = x > 0$.

Solving $x^2 + x + (1 - c) = 0$ gives $x = \frac{-1 \pm \sqrt{4c - 3}}{2}$

in which the negative solution $x = \frac{-1 - \sqrt{4c - 3}}{2}$ is to be rejected

Indeed $\frac{-1 + \sqrt{4c - 3}}{2}$ is non negative only for $c \geq 1$.

To sum up:-

If $c = 0$, $x = 0$ is the only solution.

If $c \geq 1$ $x = \frac{-1 + \sqrt{4c - 3}}{2}$ is the only solution.

If $c < 1$ and $c \Leftrightarrow 0$ there is no solution.

Comment: - If $\sqrt{b}$ can denote either square root of b (contrary to the accepted convention when working with real numbers) then there are 4 solutions:
$x = \frac{-1 \pm \sqrt{4c - 3}}{2}$ and $x = \frac{1 \pm \sqrt{4c + 1}}{2}$; the last two obtained by solving

$$x^2 - x - c = 0$$

Correct solution from F. Antonuccio).

Q. 705. We call an integer n "good" if it can be represented in the form $5x + 11y$ where x and y are non-negative integers, and "bad" otherwise. (For example, all negative integers are bad, but 0 and $27 = 5 \times 1 + 11 \times 2$ are both good.)

Find a whole number c with the property that whenever c is expressed as the sum of two integers, one of them is good and the other bad. How many positive bad numbers are there?

Generalize both parts when 5 and 11 are replaced by h and k, any two relatively prime whole numbers.

Solution: Note that $m = 5\times(-2m) + 11 \times m$. If m is <u>any</u> integer it is expressable as $5x + 11y$ with x, y integers. In fact if $5x + 11y = m$ then $5(x + 11t) + 11(y - 5t) = m$ for any integer value of t. Hence for any m there are an infinite number of ways to express it in the form $m = 5x + 11y$, $x, y \in \mathbb{Z}$ Of these, we can always find exactly one, $m = 5X + 11Y$, in which X is one of $0, 1, 2, \ldots, 10$. Indeed there is a unique expression for m in this form.

[Since $5X_1 + 11Y_1 = 5X_2 + 11Y_2 \rightarrow 5(X_1 - X_2) = 11(Y_2 - Y_1)$,

$X_1 - X_2$ must be a multiple of 11.

But since X_1, X_2 are both in $\{0, 1, 2, \ldots, 10\}$ their difference is less than 11.

Hence we must have $X_1 - X_2 = 0$ etc.].

It follows from this that an integer is "good" if it is expressible in the form $5x + 11y$ where x is one of $0, 1, \ldots, 10$ and $y \geq 0$. If it is "bad" it is expressible as $m = 5x + 11y$ with x one of $0, 1, 2, \ldots, 10$, but $y < 0$.

The largest "bad" integer is found by taking $x = 10$, $y = -1$

$m = 5 \times 10 + 11 \times -1 = 39$ is the largest bad integer.

Suppose this number is expressed as $u + v$

Let $u = 5x_1 + 11y_1 \qquad 0 \leq x_1 \leq 10$

$v = 5x_2 + 11y_2 \qquad 0 \leq x_2 \leq 10$

$$u + v = 5(x_1 + x_2) + 11(y_1 + y_2) = 5 \times 10 + 11 \times (-1)$$

Since $x_1 + x_2 - 10$ is a multiple of 11 but $0 \leq x_1 + x_2 \leq 20$

we must have $x_1 + x_2 = 10$ and hence $y_1 + y_2 = -1$

One of y_1, y_2 must be a negative integer the other non-negative.

$\therefore$ One of u, v is "good", the other "bad". Hence the largest "bad" number 39 has the properties required of c. (There is no other possible choice for c:- If $m > 39$ it is "good" and $0 + m = m$ has both summands good. If $m < 39$, then $m = 39 + (m - 39)$ has both summands bad.) Now consider the twenty pairs (0, 39), (1, 38), (m, 39 - m), ... (19, 20). By the foregoing, each of these pairs contains one positive "bad" integer and every positive bad integer is contained in a pair. Hence the number of positive bad integers is exactly 20.

Assuming that, when h and k are any two relatively prime whole numbers, every integer is expressible in the form $m = h x + k y$ $(x, y \in \mathbb{Z})$, the above arguments carry through without essential change to show that:-

(i) There is a unique way of expressing m in the form $m = h x + k y$ with

$$x \in S = \{0, 1, 2, \ldots, k - 1\}$$

(ii) The largest "bad" integer is $c = h \times (k-1) + k \times (-1)$

$$= h k - h - k.$$

(iii) If $c = u + v$ then one of the integers u, v is "good" the other "bad"

(iv) The number of positive "bad" integers is $\frac{1}{2}(c + 1)$

$$= \frac{1}{2}(h k - h - k + 1).$$

<u>Q. 706</u>. The following expression is expanded out, and all the coefficients of the resulting polynomial are added together. What result is obtained?

$$(1 - 3x + 4x^2)^{12} (5 - x - 3x^3)^{10}.$$

<u>Solution</u>: If P(x) is any polynomial, $a_0 + a_1 + \ldots + a_n x^n$ the sum of its coefficients is equal to $P(1) = a_0 + a_1 \times 1 + \ldots + a_n \times 1^n$

$\therefore$ The result obtained is $(1 - 3 \times 1 + 4 \times 1^2)^{12} (5 - 1 - 3 \times 1)^{10}$

$$= 2^{12} \times 1^{10} = 4096$$

<u>Q. 707</u>. It is known that

$$2x_1 - 5x_2 + 3x_3 \geq 0$$

$$2x_2 - 5x_3 + 3x_4 \geq 0$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

$$2x_{n-1} - 5x_{n-1} + 3x_n \geq 0$$

$$2x_{n-1} - 5x_n + 3x_1 \geq 0$$

$$2x_n - 5x_1 + 3x_2 \geq 0 \ .$$

Let $x_1 = 7$. Find the numbers $x_2, x_3, \ldots, x_n$.

__Solution__: (We have corrected the more or less obvious misprint in the last inequality). Let $S = x_1 + x_2 + \ldots + x_n$.

Adding all the inequalities gives

$$2S - 5S + 3S \geq 0.$$

If any of the $\geq$ signs were replaced by $>$ we would have obtained $2S - 5S + 3S > 0$ which is obviously false. Hence we must have equality in all n relations.

The first equation can now be written

$$2(x_1 - x_2) = 3(x_2 - x_3)$$

whence $$(x_2 - x_3) = \frac{2}{3}(x_1 - x_2)$$

Similarly $(x_3 - x_4) = \frac{2}{3}(x_2 - x_3) = (\frac{2}{3})^2(x_1 - x_2)$.

and eventually $(x_n - x_1\ ) = \frac{2}{3}(x_{n-1} - x_n)$ (from the 2nd last equation)

$$= (\frac{2}{3})^{n-1}(x_1 - x_2)$$

and $(x_1 - x_2) = \frac{2}{3}(x_n - x_1)$ (from the last equation)

$$= (\frac{2}{3})^n (x_1 - x_2)$$

$\therefore$ $x_1 - x_2$ must equal 0. Thus $x_2 = x_1 = 7$

In a similar way we can show that $x_2 = x_3$, $x_3 = x_4$ etc.

$\therefore$ the only solution when $x_1 = 7$ is $x_i = 7$ for all $i = 1, 2, \ldots, n$.

(Apart from an omission in proving the uniqueness of the answer, a good solution was received from F. Antonuccio.