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PROBLEM SECTION

Solutions of the following problems will appear in the next issue of Parabola. You are invited to submit answers to one or more of them.

Q.720. When the initial digit of a whole number x is deleted, the number decreases by a factor of 13.
Find all possible values of x.

Q.721. When the initial digit of a whole number x is deleted, the number decreases by a factor k.
Find all possible whole numbers k for which there do exist solutions for x.

Q.722. A list of numbers $\{x_1, x_2, x_3, x_4, \ldots, x_n, \ldots\}$ is constructed as follows:- any four positive whole numbers less than 100 are chosen for x_1, x_2, x_3 and x_4. For $n > 4$, x_n is the number formed from the last two digits of the sum of the previous 4 numbers. e.g. the list starting {21, 73, 86, 20, ... would continue ..., 0, 79, 85, 84, 48, ... }. Is it possible that the number x_1 never occurs a second time in the list? Prove your assertion.

Q.723. A chain has N links. Seven appropriately chosen links are cut enabling the chain to be separated into pieces. If x is any whole number not exceeding N, it is possible to find some of the pieces containing altogether exactly x links.
Find the largest possible value of N.

Q.724 Let P(x) denote the polynomial

$$\binom{2k+1}{1}(1-x^2)^k x - \binom{2k+1}{3}(1-x^2)^{k-1}x^3$$
$$+ \binom{2k+1}{5}^{k-2}(1 - x^2)^{k-2} x^5 \ldots + (-1)^k \binom{2k+1}{2k+1} x^{2k} + 1.$$

(k denotes any positive integer. The notation $\binom{n}{r}$ denotes the binomial coefficient for which ${}^nC_r$ is also sometimes used.)

Use de Moivres theorem to show that

$$P(\sin\alpha) = \sin(2k+1)\alpha.$$

Deduce that $P(x)$ factorizes as follows:-

$$P(x) = (-1)^k\, 2^{2k}\, x \left(x^2 - \sin^2 \frac{\pi}{2k+1}\right)\left(x^2 - \sin^2 \frac{2\pi}{2k+1}\right)\left(x^2 - \sin \frac{3\pi}{2k+1}\right) \,..$$

$$\ldots \left(x^2 - \sin^2 \frac{k\pi}{2k+1}\right).$$

Q.725. Assuming the result asserted in Qu 724 show that for any positive integer k

(i) $\sin \frac{\pi}{2k+1} \cdot \sin \frac{2\pi}{2k+1} \cdot \sin \frac{3\pi}{2k+1} \ldots\ldots \sin \frac{k\pi}{2k+1} = \frac{\sqrt{2k+1}}{2^k}$

(ii) $\operatorname{cosec}^2 \frac{\pi}{2k+1} + \operatorname{cosec}^2 \frac{2\pi}{2k+1} + \ldots + \operatorname{cosec}^2 \frac{k\pi}{2k+1} = \frac{2}{3} k\,(k+1)$

and simplify

(iii) $\cot^2 \frac{\pi}{2k+1} + \cot^2 \frac{2\pi}{2k+1} + \ldots + \cot^2 \frac{k\pi}{2k+1}$

Q.726. Using (ii) and (iii) in Q. 725 deduce that if $S_k = \frac{1}{1^2} + \frac{1}{2^2} \ldots + \frac{1}{k^2}$

$$\frac{\pi^2}{6}\left[1 - \frac{6k+1}{(2k+1)^2}\right] < S_k < \frac{\pi^2}{6}\left[1 - \frac{1}{(2k+1)^2}\right]$$

and find the "limit sum" of the infinite series

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots + \frac{1}{k^2} + \ldots$$

Q.727. When a certain polynomial, $P(x)$, is divided by $(x - 3)$ the remainder is 5. When $P(x)$ is divided by $(x+1)$ the remainder is -3. Find the remainder when $P(x)$ is divided by $x^2 - 2x - 3$.

Q.728. Find all solutions of the simultaneous equations

$$x_1 + x_3 = x\,x_2\,;\quad x_2 + x_4 = x\,x_3\,;$$

$$x_3 + x_5 = x\,x_4\,;\quad x_4 + x_1 = x\,x_5;\quad x_5 + x_2 = x\,x_1.$$

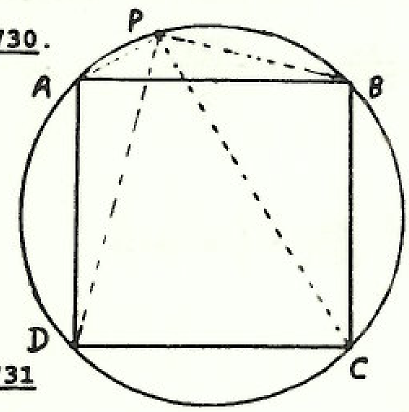
Q.729 A set of cups is arranged in a rectangular array of m rows and n columns, and a random number of beans is placed in each cup (no cup being left empty). The following operations are permitted.

(1) One bean is taken from <u>every</u> cup in a row. (This is not possible obviously if some cup in the row is already empty).

(2) The number of beans in every cup in any column is doubled.

Is it always possible to perform these operations repeatedly in such a way that all cups are eventually emptied?

Q.730.

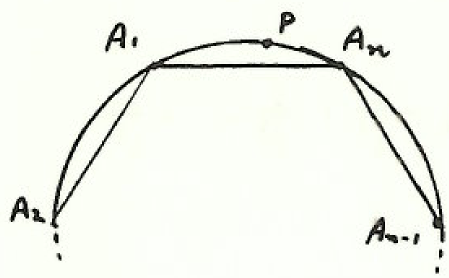


In the figure, A B C D is a square, and P is a point on the arc AB of the circumcircle. The distances of P from A, B, C and D are denoted by a, b, c, and d respectively. Show that

$$(\sqrt{2} + 1)\,(a + b) = d + c$$

and that $a - b = (\sqrt{2} + 1)(d - c)$

Q.731 Generalize the first result in Q. 730:-



Let P lie on the arc $A_1\,A_n$ of the circumcircle of a regular polygon $A_1\,A_2\,\ldots\,A_n$. Let $x_1, \ldots x_n$ denote the distances of P to $A_1, \ldots A_n$ respectively.

show that $x_2 + x_3 + \ldots + x_{n-1} = \dfrac{\cos\frac{\pi}{n}}{1-\cos\frac{\pi}{n}}\,(x_1 + x_n)$