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PROBLEM SECTION

You are probably aware that the International Mathematical Olympiad was held in Australia for the first time in this, our bicentenary year. Q.744 and Q.746-748, although not identical to, are closely related to four of the six questions that were set; and Q.745 similarly owes its inspiration to a question which was proposed, but not used. Q.744-747 have been modified to incorporate the numbers 1788, 1988 and 200 in some manner.

Q.749 (ii) is a result known as Morley's Theorem, which is one vault of elementary plane geometry which was apparently overlooked by the ancient Greeks.

I am indebted to Dr. P. Donovan for the ideas behind Q.751 and Q.752.

Q.744 Define $f(x) = \sum_{k=114}^{184} \frac{k}{x-k}$

Describe the graph of $f(x)$ and observe that if c is any positive number, the set $\{x:f(x)>c\}$ is the union of 71 interals. Let $L(c)$ denote the sum of the lengths of these intervals.

Show that $L\left(\frac{1988}{336}\right) = 1788$, and $L\left(\frac{1788}{336}\right) = 1988$.

Q.745 A function $f(n)$ is defined for positive integers n in such a way that

$f(1) = 1$, $f(2) = 2$, and

if $3^m \le n < 3^{m+1}$ for a non-negative integer m then

$f(3n+k) = k\,3^m + f(n)$

for $k = 0, 1,$ or 2.

For how many values of n between 1788 and 1988 is $f(n) = n$?

Q.746 For any positive integer n let $g(n)$ denote the number of coefficients in the expansion of $(1+x+x^2+x^3)^n$ which are odd numbers.

Show that $g(1788) = g(1988)$.

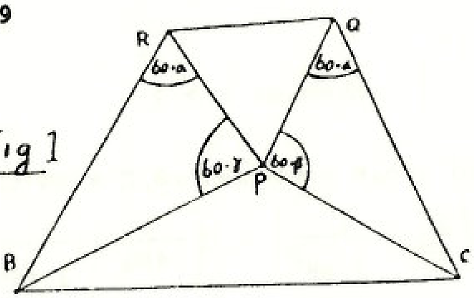
Q.747 The triangle ΔABC has side lengths $AB = 25$, $AC = \frac{65}{3}$, $BC = \frac{70}{3}$.

The internal and external bisectors of $B\hat{A}C$ cut BC at points E, F respectively, and D is the foot of the perpendicular from A to BC. The line passing through the incentres of triangles ΔADE and ΔADF cuts AE at K and AF at L.
Show that the area of $\Delta AKL = 200$.

Q.748 Let a,b be positive integers such that $q = \dfrac{a^2 + b^2}{ab + 2}$ is also an integer.
Prove that $2q$ is a perfect square.

Q. 749

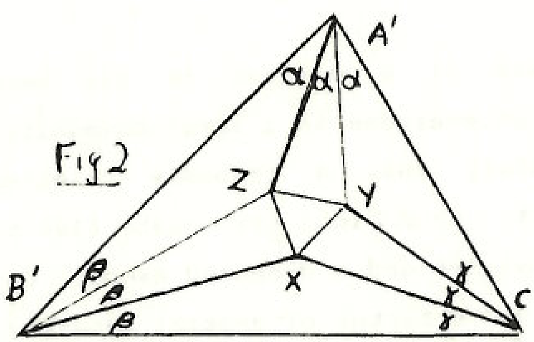
Fig 1



(i) ΔPQR is equilateral and α, β, γ are any three angles such that $\alpha + \beta + \gamma = 60^\circ$. Points B and C are constructed as in Figure 1 making $B\hat{R}P = C\hat{Q}P = 60^\circ + \alpha$, $B\hat{P}R = 60^\circ + \gamma$, and $C\hat{P}Q = 60^\circ + \beta$.

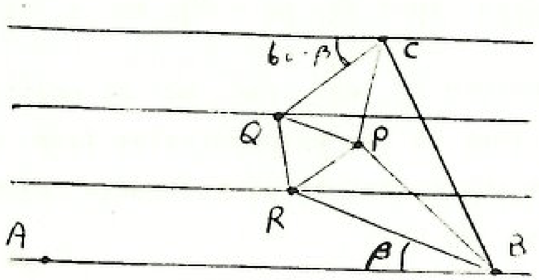
Show that $PB \sin\beta = PC \sin\gamma$, and that PB and PC are bisectors of $R\hat{B}C$ and $Q\hat{C}B$ respectively.

Fig 2



(ii) Let $A'B'C'$ be any triangle with angles 3α, 3β, 3γ. Let lines which trisect the angles intersect in pairs at points X, Y, Z as in Figure 2. Prove that ΔXYZ is equilateral.

Q. 750 A page is ruled with equally spaced parallel lines. Points A, B lie on the fourth line from the top of the page. R is a point on the next line up such that $R\hat{B}A = \beta\ (<30^\circ)$, C is the point on the top line such that $C\hat{B}A = 3\beta$. Q is the point on the second top



line (lying on the same side of BC as A and R) such that QC makes an angle $(60^\circ - \beta)$ with the ruled lines. The bisectors of $Q\hat{C}B$ and $R\hat{B}C$ intersect at P (see Figure). Prove that ΔPQR is equilateral.

Q. 751 When B. Rainy, the school genius, knocked his calculator off the desk during the maths exam, he discovered that only the algebraic operations $+, \times, -, -$ were still operational. The only exam question still to be answered required a calculation involving $\log_e 11$. Our hero recollected seeing in a calculus text the theorem

$$\log_e\left(\frac{x+1}{x-1}\right) = 2\left[\frac{1}{x} + \frac{1}{3x^3} + \frac{1}{5.x^5} + \dots\right] \quad \text{whenever } x > 1.$$

After a little calculating he found whole numbers A,B,C such that

$$\log_e 11 = A\left(\frac{1}{23} + \frac{1}{3.23^3} + \frac{1}{5.23^5}\right) + B\left(\frac{1}{65} + \frac{1}{3.65^3}\right) - C\left(\frac{1}{485} + \frac{1}{3.485^3}\right)$$

to eight decimal places. He evaluated this on his calculator and completed the examination, obtaining his usual 100%.

Can you find A,B,C and calculate $\log_e 11$?

Q. 752 In order to locate all the carriers of a disease it has been decided to perform blood tests on everyone in a large community. It is known that the probability that a randomly selected individual has the disease is 1%. Each blood test costs \$100 to perform, but the test is very sensitive and a diseased sample can be detected even if diluted by a factor of several thousand. Instead of just testing each sample separately, it would obviously be cheaper to mix small portions of each of a batch of samples together and test the mixture. For example, if batches of 4 samples were mixed, most of the batches would yield a negative result. Any batch which tested positive would necessitate further tests on the remaining portions of the 4 samples to determine the

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