PROBLEM SECTION

You are invited to submit solutions to any or all of the following problems, accompanied by your name, school and year or form. Solutions of these problems will appear in the next issue of Parabola; your solution(s) may be used if they are received in time.

Q.773 Prove that $\tan 36° \times \tan 72° = \sqrt{5}$.

Q.774 Let x be a whole number none of whose digits is 0. Let $d(x)$ be the highest common factor of all the numbers obtainable by rearranging the digits of x.

e.g. if $x = 468$, $d(x) = 18$ since this is the h.c.f. of $\{468, 486, 648, 648, 684, 846, 864\}$.

If the digits of x are not all the same, find the largest possible value of $d(x)$, and the smallest value of x for which $d(x)$ has that maximum value.

Q.775 Let n be any whole number. Find a perfect square with $2n$ digits all less then 7, with the property that if every digit is increased by 3 the resulting number is another perfect square.

[For example, if $n = 1$, 16 is the required number, since 49 is also a square.]

Q.776 A list of numbers $a_0, a_1, a_2, \cdots, a_n, \cdots$ has the following properties

(i) $a_0 = 1$

(ii) $a_{n+2} = a_n - 2a_{n+1}$ for all $n \geq 0$.

(iii) a_n remains positive however far the list is extended. (i.e. $a_n > 0$ for all $n \in \mathbb{N}$)

Show that there is only one such list, and find it.

Q.777 If m is any given positive integer let N_m be the number of different solutions of

$$w + x + y + z = m$$

where each of the unknowns w, x, y, z denotes a non-negative whole number.

[e.g. $N_1 = 4$ since there are four solutions of $w + x + y + z = 1$; viz

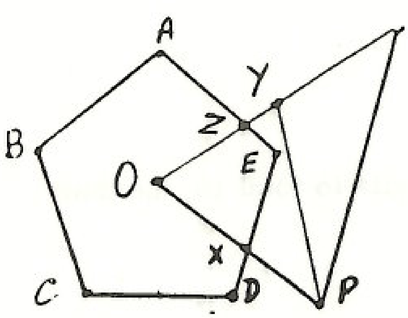
$$(w, x, y, z) = (1,0,0,0) \text{ or } (0,1,0,0), \text{ or } (0,0,1,0), \text{ or } (0,0,0,1)]$$

Prove that N_m is also the number of different ways of arranging $m\,1's$ and three 0's in a row, and deduce that $N_m = C(m+3,3) = \frac{(m+3)(m+2)(m+1)}{6}$.

Q.778 A whole number less than 10000 is chosen randomly. (Zero is a possible choice). What is the probability that the sum of the digits is less than 20?

Q.779 Let a and b be positive integers. When $a^2 + b^2$ is divided by $a + b$ the quotient is q and the remainder is r. Find all pairs (a, b) such that $q^2 + r = 1989$.

Q.780 In the accompanying figure, $ABCDE$ is a regular pentagon whose centre is at O.



The following lengths are given: $\overline{DX} = 3cms$, $\overline{XE} = 7cms$; $\overline{OP} = \overline{PY} = \overline{YQ} = 20cms$. Also $\overline{OQ} = \overline{PQ}$. Calculate the area of the quadrilateral $OXEZ$.

Q.781 A set S consists of 100 different positive whole numbers, the largest of which is x. The sum of three different numbers chosen from S is never equal to a fourth number in S. Find the smallest possible value of x. Exhibit a set S having this value of x, and prove that no smaller value is possible.

CROSS NUMBER PUZZLE SOLUTION

[1] 1	[2] 6	[3] 9	[4] 7
[5] 3	4	3	1
[6] 2	[7] 6	[8] 2	7
5	[9] 5	1	2

If the digits of a whole number N add to 15, it is divisible by 3, but not by 9. Hence N is not a perfect cube, nor the product of a cube by a two digit prime. Thus neither $3d$ nor $4d$ is a perfect cube. Since $1d$ is a prime with two multiples ($6a$ and $7d$) less than 100, it does not exceed 31. Hence $1a \neq 729$, so $1a$ is not a cube as well as a square. There are only two different 2 digit cubes, so there must be 2 cubes with 3 or more digits in the puzzle. Hence $5a$ and $9a$ must both be cubes since we have ruled out all the others. The first digit of $5a$ is not 2 or 5 since $1d$ is prime. If $5a$ is 125 or 729, $2d$ ends with 2, so $7d$ ends with 3, and $9a$ would be 343. But this is impossible since $2d$ is to be a factor of $9a$; 343 has no factor ending in 2.

$\therefore$ We must have $5a = 343$. Since $7d$ ends in 5, $7d = 5 \times 1d$

$\therefore$ $1d = 13$, $7d = 65$, $2d = 64$, $1a = 169$, $9a = 512$. For the digits to total 15, the third digit of $3d$ must be 2, $3d = 9321$.

$\therefore$ $4d = 3d \div 13 = 717$. To give a third square, $6d = 25$.