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History of Mathematics
Pythagoras and Theano

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Some years ago, I wrote a biography of Hypatia of Alexandria, the first female mathematician of whose life and work we have a good body of reliable evidence. (See my *Hypatia of Alexandria: Mathematician and Martyr,* Prometheus Press, 2007.) But I kept coming across references to an earlier woman mathematician, Theano. For example, Nicephorus Gregoras, a 14th Century Church historian, described a woman, whose mathematical ability he wished to praise, as "another Theano or another Hypatia". I decided to see what I could learn of this other (earlier) female mathematician. So in this column and my next I want to share what I *did* learn, to indicate the difficulties in discovering much more, and to use the opportunity to explore some interesting Mathematics along the way.

We learn that Theano was an associate (very possibly the wife) of a much better-known mathematician, Pythagoras, and in order to find out about Theano, we need first to learn of Pythagoras. Now Pythagoras himself, although his *name* is of course extremely familiar, is actually a rather shadowy figure. There is an accessible article on the Wikipedia website, and you will find another at

http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Pythagoras.html

These are good introductions, and they attempt to tell the history as best as possible, but they also describe the difficulties involved. In fact, if you look at them, you will be struck immediately by the fact that they actually contradict one another in some places. This is par for the course! A much fuller account is available in the book *Pythagoras: A Life* by Peter Gorman (Routledge & Kegan Paul, 1979). Gorman describes one attempt at completeness in researching Pythagoras's life as "a titanic struggle, since perhaps no other person is mentioned more frequently in the ancient authors".

Three of these "ancient authors" wrote actual biographies, but many many others have something or other to say about him. A more recent (1990) work than Gorman's (Luis Navia's *Pythagoras: An Annotated Bibliography*; New York: Garland Publishing Co.) does provide a list of 41 works that between them contain all the ancient writings about and by the Pythagoreans.

The three biographers were Porphyry, Iamblichus and Diogenes Laertius. Of these, the first two were philosophers of the 3rd Century CE, and the third was a biographer of roughly the same period. I was able to get hold of the second and third of these in English translation without much difficulty; it took rather more effort to get hold of the first, but eventually, after some hassle, I was able to borrow a French translation. The first and third of these biographies are rather short: only 31 pages and 20 pages respectively. The second is much fuller: in English translation, it runs to 139 pages.

A besetting difficulty, however, is that these ancient sources also often contradict one another or even themselves, and furthermore can also be at odds with other authors' statements. However, some outlines are generally agreed. Pythagoras lived from *about* 580BCE to *about* 500BCE (although these dates are very uncertain). He was described as a "philosopher", a word that he may himself have coined. In its original Greek, it means a "lover of wisdom", and this description is perhaps the best one to keep in mind. He is known to have founded a society dedicated to this purpose, and to have attracted many disciples, women as well as men. Among these was Theano.

Theano is described by Porphyry as "especially famous". He mentions her in two places. In the first passage, he writes: "According to [some sources], Theano, daughter of Pythonax, and a Cretan by birth, gave Pythagoras a son Telauges and a daughter Myia." Later he has: "All this [achievement] earned [Pythagoras] great renown, which attracted to him many followers from that town [Croton in southern Italy], not just men but also women, at least one of whom, Theano, made a great name for herself."

Diogenes Laertius mentions Theano as Pythagoras's wife. "Pythagoras had a wife, Theano by name, daughter of Brontinus of Croton, although some call her Brontinus's wife and Pythagoras's pupil."

Iamblichus mentions her twice. In one passage, he calls her Pythagoras's wife. Later on, he lists her as the sixth in a list of names of seventeen female Pythagoreans, but without giving her any further particular prominence; here he says she was the wife of Brontinus, whom other sources list as her father. [I rather think that the manuscript of Iamblichus's *Life* may have been miscopied at this point, and the word "wife" been wrongly used instead of "daughter" when her relationship with Brontinus is introduced. It should be borne in mind that the manuscripts that have come down to us from such ancient sources are not the originals, but more likely hand-written copies of copies of copies, etc. I also find it surprising that this *real* inconsistency in the current version of Iamblichus's text is hardly ever remarked, whereas the relation of stories relevant to my next column are almost universally found wanting, when in that case poor old Iamblichus was merely passing on different versions of essentially the same story!]

A further difficulty that all biographers face stems from the fact that the members of the society took a vow of silence, although it remains a matter of dispute quite how complete this was.

The full members of the society were called the *mathematikoi* (which is the origin of our word "mathematics"); those outside the inner circle of full members were the *akousmatikoi*. These could listen in on the discussions of the *mathematikoi*, but could not themselves take part. It seems that the discussions of the *mathematikoi* were witnessed by large audiences of *akousmatikoi* ("listeners": compare our word "acoustics"), and very possibly members of this larger group did not feel the need to remain silent.

It is also often asserted that many of the discoveries made by the society were in fact joint efforts, but were all respectfully attributed to Pythagoras himself. Again, this is a matter of some dispute. It does, however, make for difficulties in deciding quite what it was that Pythagoras himself achieved.

Rather sadly, Gorman's biography says very little of Pythagoras's Mathematics (and nothing at all of Theano's). For that, we need to look elsewhere. There is, however, a useful article on precisely this question. Its author was the Leningrad-based Leonid Zhmud, who wrote mainly in Russian but whose research was also the subject of a paper written in English and published in the journal *Historia Mathematica* in 1989.

According to Zhmud, from the early years of the 20th Century until into the 1980s, the climate of opinion "[cast doubt] on the reality of Pythagoras's mathematical activity". Rather it was widely held that "Pythagoras was first [and foremost] a religious figure and is presented as such in the early sources", that when it came to Mathematics, he was not so much an original thinker as a transmitter of "Egyptian (or Babylonian) tradition", and that the main advances in early Greek Mathematics came after his death.

Zhmud counters that, although Pythagoras most certainly *was* interested in religious matters, he is also presented by the earliest ancient authors as "a rational thinker, a scientist, and a person of vast knowledge". He also rejects the idea that his Mathematics was derived from other sources. [However, on this more later.] As to the third objection, Zhmud counters that although what we have today of the earliest Greek Mathematics comes from later times, these developments must have sprung from roots already planted.

He goes on to consider eight pieces of evidence pointing to original mathematical activity on Pythagoras's part. All come from the earliest available sources (4th Century BCE). Here are the eight.

(i) An Athenian named Isocrates claimed that Pythagoras, borrowing from the Egyptians, espoused a philosophy comprising "geometry, arithmetic and astronomy".

(ii) "Plato's disciple Xenocrates ... testifies to Pythagoras's discovery of the numerical expression of harmonic intervals", thus initiating a mathematical approach to music.

(iii) Aristotle wrote that "Pythagoras ... first devoted himself to the study of mathematics, in particular of numbers, but later [developed other interests]".

(iv) Aristotle also wrote that "the so-called Pythagoreans were the first to engage in the study of the mathematical sciences, greatly advancing them ... [they] began to consider their principles [i.e. those of the mathematical sciences] as the principles of all things ... ".

(v) "Aristoxenus, a disciple of the last Pythagoreans, ... considered that 'Pythagoras honored the study of numbers more than anyone. He made great advances in it, withdrawing it from the practical calculations of merchants and likening all things to numbers'."

(vi) The mathematician Proclus, in his commentary on Euclid, and quoting the earlier philosopher Eudemus, had this to say, "Pythagoras transformed the philosophy of geometry, making it a form of liberal education, considering its principles abstractly, and examining the theorems immaterially and intellectually. He discovered the theory of proportionals and the construction of cosmic bodies".

(vii) "Diogenes Laertius relates that a certain Apollodorus the Calculator credits Pythagoras with the proof of the theorem that the squares of the sides of a right-angled triangle are equal to the square of the hypotenuse."

(viii) "[Probably also basing their statements on the writing of the earlier philosopher Eudemus], Hero of Alexandria ... and after him Proclus ... attribute to Pythagoras the method of determining the sides of a right-angled triangle (the Pythagorean triplets)".

Zhmud then goes on to consider what these eight pieces of evidence lead us to infer about the Mathematics Pythagoras and his circle discussed and developed. He lists: (1) the theory of proportionals, (2) the theory of odd and even numbers, (3) the Pythagorean theorem, (4) the method of determining the Pythagorean triplets, (5) the construction of the first two regular polyhedra. [And we may remark in passing that the above quotations, especially ## 5, 6, are most apt descriptions of *pure* Mathematics as a discipline in its own right.]

But now to examine in detail the five areas that Zhmud catalogues. Look first at the theory of proportionals, and its relation to music. An ancient instrument, the *monochord*, may very well have been familiar to the Pythagoreans. In its most basic form, a monochord is a single string is stretched over a sound box and fixed at both ends. A movable "bridge" can be manipulated to alter the effective length of the string as it is set vibrating by plucking it. With the bridge absent, the string will vibrate at a given frequency, the *fundamental*. If we now insert the bridge and position it at the midway point of the string, the effective length is halved, because the plucking causes only one half of the string to vibrate, but the frequency of the vibration is doubled. The note then heard is one *octave* above the fundamental.

More generally, if l is the effective length and f is the frequency of vibration, then f is proportional to $1/l$. Other divisions of the original string are possible, each corresponding to a frequency, i.e. a note. Often a monochord is provided with a second string, this one without a bridge. By sounding the two strings together, one can check whether the two notes sound harmonious together. As a rough rule of thumb, this is best achieved when the two frequencies are related as a ratio of small integers, and this implies that the effective lengths must also be related as the ratios of small integers.

According to Zhmud, Pythagoras discovered the division of a string in the ratios 1:2, 2:3, 3:4 all produced pleasing harmonies. Musicologists name these as, respectively, the *octave*, the *fifth* and the *major fourth*. They correspond, again respectively, to a doubling of the fundamental frequency, to a frequency of $\frac{3}{2}$ of the fundamental and to a frequency of $\frac{4}{3}$ of it. A fifth will then be generated if we place the bridge at a point $\frac{2}{3}$ of the way along the string, and a fourth if it is placed $\frac{3}{4}$ of the way.

If we consider two different frequencies of vibration, f_1, f_2 , say, then their (arithmetic) mean $\dfrac{f_1 + f_2}{2}$ will be another frequency, and this will often be harmonious with the original two if these are related as the ratio of small integers. But the arithmetic mean corresponds to a bridge position that is discovered by using the relation: f proportional to $1/l$. So the corresponding bridge position will be given by $\frac{1}{2}\left(\frac{1}{l_1} + \frac{1}{l_2}\right)$. The reciprocal of this last number lies between l_1 and l_2. [You might like to prove this as an exercise.] This new number is then another "mean" and because of its musical associations it was referred to as the "harmonic mean", a name it still bears today.

There is, of course, a lot more that can be said on this subject; whole books have been written about it. But enough has been presented here to show the mathematical flavor of such work. Let us now turn to the second entry of Zhmud's list: the theory of odd and even numbers.

Here Zhmud is concerned to show that some of the Pythagorean work on numbers became part of the Greek mathematical tradition and some of it may even be found, almost unchanged, in Euclid's *Elements*. (Euclid lived about 250 years later than Pythagoras.)

By no means all of Pythagoras's notions on number stand scrutiny today. (He considered odd numbers to be male, and even numbers female, and there are other such identifications that strike modern minds as merely silly. However, in a culture of rampant Freudianism, at least one eminent 20th Century mathematician, whose memory I will spare by not naming him, wrote in quite similar fashion!) Moreover, Pythagoras did not see 1 as a number at all, and here there was some firmer basis. 1 was seen as the unit of measurement; *numbers* were produced by using the unit to measure other quantities. I am reminded of the debates over whether 0 should be regarded as a natural number, and whether 1 should be classed as prime. (See my column in *Parabola*, *Volume* **45**, *No*. **2**.) Today we *do* of course regard 1 as a natural number, and so have moved away from this Pythagorean viewpoint. However the best of Pythagoras's work on number still commands interest today. There was, for instance, focus on the relation between number and geometry: triangular numbers, square numbers and the like.

Again, he became interested in "amicable" or "friendly" numbers. Two numbers are "amicable" if the proper divisors of the first add up to the second, and vice versa. (When we say "proper divisors" we mean that the number itself is not counted as one of them.) The first known such pair is (220, 284). The proper divisors of 220 are 1, 2, 4, 5, 10, 11, 20, 22, 44, 55, 110; these numbers add up to 284. The proper divisors of 284 are 1, 2, 4, 71, 142 and these numbers add up to 220. Iamblichus explicitly attributes the discovery of amicable numbers to Pythagoras. It is not clear how many such pairs he might have known: the next three such pairs are (1184, 1210), (2620, 2924), (5020, 5564). Many more have been discovered, up to truly large pairs. It is conjectured that there are infinitely many such pairs, but no proof has yet been found.

More problematical is the relation of Pythagoras to the so-called "perfect numbers". These are numbers whose proper divisors add up to the number itself. The smallest such number is 6 [the proper divisors are 1, 2 and 3, for which $1 + 2 + 3 = 6$]. There are other perfect numbers: 28, 496, 8128, 33550336, and even larger ones. Up to 8128, this list was known to the ancient Greeks, possibly even to Pythagoras and his society. Zhmud however regards this as rather doubtful. The topic however retains its interest for today's mathematicians. There are in fact several unsolved problems concerning it. We do not yet know if there are any odd perfect numbers, nor whether there are infinitely many perfect numbers (as of mid-2009, only 47 were known). We *do* know that all even perfect numbers are triangular (i.e. of the form $\frac{n(n+1)}{2}$) and that the decimal expansions of all such numbers must end in either a 6 or an 8, but there is a lot still unknown.

Zhmud's 3rd and 4th suggestions are closely related. Let us begin with the question of whether Pythagoras's theorem is in fact due to Pythagoras. Start with Gorman's view, which is in the negative. "In the modern world [Pythagoras] is chiefly remembered for an achievement almost certainly not his: the theorem concerning the hypotenuse and sides in a right-angled triangle." "[His] famous theorem ... was certainly known in Babylon long before his birth."

Gorman and Zhmud rely on the same sources, but they interpret them very differently. Although Diogenes Laertius wrote over 700 years after Pythagoras's death, he was quoting from a much earlier source (how much earlier is still a matter of dispute). He goes on the give a couple of lines of verse:

"When Pythagoras that famous figure found
A noble offering he laid down!"

This epigram is also to be found in the writing of Cicero and five other ancient authors (including Porphyry). The fact that so many different people found it worth quoting is interpreted by Zhmud as evidence of the importance attached to this result.

Now it is certainly true that Pythagoras's theorem was known to the Babylonians long before Pythagoras was born; it may even be the case that he learned of it from them or else from some others. So, if he is to be celebrated in this connection, it must be for some major contribution to the understanding of the result. The obvious possibility is that what he supplied was a proof. Although the Babylonians were obviously aware of the result, they may nonetheless have been unable to *prove* it, or very possibly were unaware of any need for a proof. Quite possibly all they did was notice a pattern, which is not the same thing. If Pythagoras *did* actually provide a proof (i.e. an explanation saying *why* the pattern holds), then this would indeed be cause for celebration! The reference to a "figure" is somewhat suggestive. (See my column in *Parabola*, *Volume* **45**, *No*. **1**). (It should however be noted that if Pythagoras *did* produce a proof, that proof has been lost. The earliest known proof is that in Euclid's *Elements* and that is widely believed to by Euclid himself.)

All in all this line of argument is speculative, and far from compelling; nonetheless, it contains a good deal of plausibility.

When we turn to the question of the Pythagorean triplets, again we find the Babylonians getting there first. A Pythagorean triplet is a set of three integers $(a,\ b,\ c)$ such that $a^2 + b^2 = c^2$. A cuneiform tablet, known as Plympton 322 and dating to many centuries before Pythagoras, lists 15 such triplets, some of them quite large. [The largest is $(12709,\ 13500,\ 18541)$.] Up till quite recently, this was seen as evidence that the ancient Babylonians had found the formula for generating such triplets, and I would guess that there are quite a lot of people who still hold this view. On the other hand, there is recent research that tends to cast some doubt on this interpretation.

However, let us take time out to see how such triplets are generated. Because $(u^2 - v^2)^2 + (2uv)^2 = (u^2 + v^2)^2$, we may always generate such triplets by setting $a = (u^2 - v^2)\,,\ b = 2uv,\ c = (u^2 + v^2)$, where $u,\ v$ are natural numbers, with $u > v$. It may also be shown that this rule generates *all* the Pythagorean triplets, although I will not stop to prove this here.

The historian of Mathematics Sir Thomas Heath, although he is one of those that Zhmud thinks sells Pythagoras's mathematical attainments short, still commands respect today. He thinks that Pythagoras was certainly aware of the case $v = 1$, but possibly no more. Much of Zhmud's analysis essentially duplicates Heath's at this point.

Zhmud actually says very little on this question, and almost none at all on his final point: the construction of the first two regular polyhedra. The regular polyhedra are probably what Eudemus meant when he referred to " the construction of cosmic bodies". There are exactly five of them: in order:

(i) The (regular) tetrahedron, with 4 equilateral triangular faces

(ii) The cube, with 6 square faces

(iii) The (regular) octahedron, with 8 equilateral triangular faces

(iv) The (regular) dodecahedron, with 12 regular pentagonal faces

(v) The (regular) icosahedron, with 20 equilateral triangular faces.

By "construction", is meant (essentially) a proof that such solids can actually exist. The final book of Euclid's Elements (Book XIII) is devoted to exactly this question. Why Zhmud restricts Pythagoras's achievement to the first two cases is unclear to me. However, I will take up these matters again in my next column.

But now the time has come to return to Theano. She was clearly one of the mathematikoi, and so she would certainly have been aware of all the Mathematics just discussed, but it is difficult reliably to go beyond this general remark. Of the three biographers, Diogenes Laertius is the most helpful in this regard. He says (as I quoted in part before): "Pythagoras had a wife, Theano by name, daughter of Brontinus of Croton, though some call her Brontinus's wife and Pythagoras's pupil. He had a daughter Damo ... [to whom he entrusted] the custody of his memoirs. ... . They [Pythagoras and Theano] also had a son Telauges, who succeeded his father ... . Telauges wrote nothing, so far as we know, but his mother Theano wrote a few things."

There is a list of these "few things" in an ancient encyclopedia known as the *Suda Lexicon*. This is a much later source (tenth century CE) so it comes well over a thousand years after the events we are discussing here. Nonetheless, we have nothing better to go on. An English paraphrase of the relevant (brief) entry is provided by Jane

McIntosh Snyder in her book *The Woman and the Lyre* (Southern Illinois Press, 1989). Four works are listed: *Concerning Pythagoras*, *For Hippodamus of Thurii (Concerning Excellence)*, *Advice for Women* and *Sayings of the Pythagoreans*. Snyder then goes on to say that "[other] titles are attributed to her as well", but rather maddeningly she gives no further details. She does go on to quote from several of Theano's letters that have survived, but none of their subject-matter is mathematical. Another source (M. Meunier's *Femmes pythagoriciennes*) prints French translations of ten letters attributed to Theano, alomg with some lines quoted from her work. Again, sadly, none of this is mathematical; one quote refers to Mathematics, but only in a general way.[1]

Sadly, this is really as far as we can go with any reliability. Other claims are made for Theano's work. One such claim is that Theano continued to run the Pythagorean society after Pythagoras's death. We have no real evidence for this, and quite a lot against it. Diogenes Laertius explicitly states that it was Pythagoras's son Telauges who took over. Iamblichus however has a different story. He lists Pythagoras's successor as Aristaeus, and states that after Pythagoras's death, Aristaeus married Theano, whom at this point he describes as Pythagoras's widow, and that when Aristaeus himself relinquished the leadership, Pythagoras's son, Mnesarchus, took it up.

Although there is no real agreement about who it was who succeeded Pythagoras, the most detailed sources say quite clearly that it was not Theano!

Another claim made for Theano is that she wrote a work on the Golden Mean, that is to say the number $\frac{1+\sqrt{5}}{2}$. There is a certain plausibility to this idea, but no more than that. In my next column, I will discuss the interest the Pythagoreans may have had in this number, but this claim for Theano is really quite baseless. Every account I have found of Theano's connection to the Golden Mean, where it gives any detail at all, derives from a passage in Lynn Osen's book *Women in Mathematics* (MIT Press, 1975). Regrettably, this is an appallingly bad book, utterly and completely unreliable. When it first appeared, there was little available work on her subject, so that it commanded interest on that account. Had it been a better work, it would have been a most valuable contribution to scholarship; as it is, however, it does not even merit the name, and its continued influence is to be deplored. The otherwise good quality entry on Theano in the Wikipedia errs by following Osen in this baseless claim, and so (sadly) do several other sources that should have been more wary!

I will return to the Pythagorean theme in my next column, and will have a lot more to say on the Golden Mean.

[1]An interesting side-issue arises when we compare Theano with Hypatia. We have a few pages of Theano's actual writing, probably none of Hypatia's. Nevertheless, we know a lot more about Hypatia than we do about Theano!