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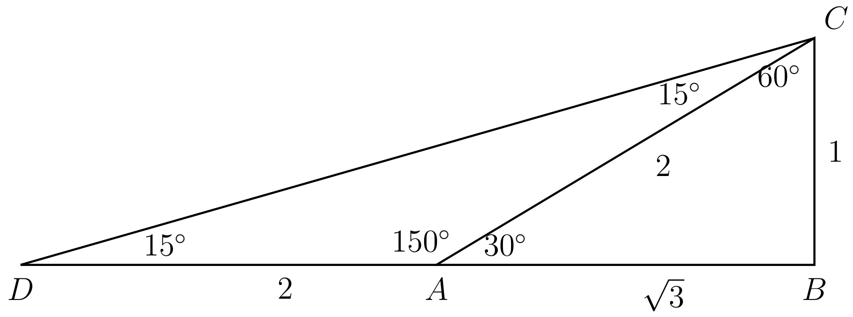
Solutions to Problems 1311-1320

Q1311 Prove that $\tan 75° - \tan 60° = 2$ using purely geometrical arguments.

ANS: (Correct solution by J.C. Barton, Victoria)



Draw a right-angled triangle ΔABC with angles and side lengths as indicated on the figure. Produce BA to D such that $DA = AC$. Then ΔACD is isoceles with angles as indicated. Hence $\langle DCB = 75°$. Now

$$\tan 75° = \frac{DB}{BC} = \frac{2+\sqrt{3}}{1} = 2 + \sqrt{3}$$
$$\tan 60° = \frac{AB}{BC} = \frac{\sqrt{3}}{1} = \sqrt{3},$$

implying

$$\tan 75° - \tan 60° = 2.$$

Q1312 Three right-angled triangles have integral side lengths. The side lengths of the first and second triangles are, respectively, 5, 12, 13, and 8, 15, 17 (units). Given that the length of the hypotenuse of the third triangle is 221 units, find the other two lengths.

ANS: (Correct solution by J.C. Barton, Victoria)

Since $13^2 = 5^2 + 12^2$, $17^2 = 8^2 + 15^2$ (by Pythagoras), and $221 = 13 \times 17$ we have

$$221^2 = 13^2(8^2 + 15^2) = 104^2 + 195^2$$

or

$$221^2 = 17^2(5^2 + 12^2) = 85^2 + 204^2.$$

Thus the other two sides may be 104 and 195 or 85 and 204. There are other solutions.

Q1313 This year (2009), November 13th falls on Friday. Explain why every year must have at least one Friday the thirteenth. What is the largest number of these days that can fall in a year?

ANS: If a month has a Friday the thirteenth then that month starts on a Sunday. Thus the question is the same as *"Explain why every year must have at least one Sunday the first. What is the largest number of these days that can fall in a year?"*

Recall that January, March, May, July, August, October, December have 31 days, February has 28 days (ordinary year) or 29 days (leap year), and the rest have 30 days. We number the months from January to December by 1 to 12, and days from Sunday to Saturday by 1 to 7. Note that $28 \equiv 0 (\text{mod } 28)$, $29 \equiv 1 (\text{ mod } 28)$, $30 \equiv 2 (\text{mod } 28)$ and $31 \equiv 3 (\text{mod } 28)$. Using modular arithmetic, we can construct Tables 1 and 2 which answer the questions. There is at least one and at most three days numbered by 1 in each row.

Month	1	2	3	4	5	6	7	8	9	10	11	12
Monday	1	4	4	7	2	5	7	3	6	1	4	6
Tuesday	2	5	5	1	3	6	1	4	7	2	5	7
Wednesday	3	6	6	2	4	7	2	5	1	3	6	1
Thursday	4	7	7	3	5	1	3	6	2	4	7	2
Friday	5	1	1	4	6	2	4	7	3	5	1	3
Saturday	6	2	2	5	7	3	5	1	4	6	2	4
Sunday	7	3	3	6	1	4	6	2	5	7	3	5

Table 1: First days of months in an ordinary year

Month	1	2	3	4	5	6	7	8	9	10	11	12
Monday	1	4	5	1	3	6	1	4	7	2	5	7
Tuesday	2	5	6	2	4	7	2	5	1	3	6	1
Wednesday	3	6	7	3	5	1	3	6	2	4	7	2
Thursday	4	7	1	4	6	2	4	7	3	5	1	3
Friday	5	1	2	5	7	3	5	1	4	6	2	4
Saturday	6	2	3	6	1	4	6	2	5	7	3	5
Sunday	7	3	4	7	2	5	7	3	6	1	4	6

Table 2: First days of months in a leap year

Q1314 Prove that in an acute-angled triangle ABC there holds

$$\tan^n A + \tan^n B + \tan^n C \geq 3 + \frac{3n}{2},$$

where $n = 0, 1, 2, 3, \ldots$.

ANS: Since ΔABC is acute-angle, there hold $\tan A$, $\tan B$, $\tan C > 0$. Since $C = \pi - (A+B)$ there holds

$$\begin{aligned}
\tan A + \tan B + \tan C &= \tan A + \tan B - \tan(A+B) \\
&= \tan A + \tan B - \frac{\tan A + \tan B}{1 - \tan A \tan B} \\
&= (\tan A + \tan B)\left(1 - \frac{1}{1 - \tan A \tan B}\right) \\
&= -\frac{\tan A + \tan B}{1 - \tan A \tan B} \tan A \tan B \\
&= -\tan(A+B) \tan A \tan B \\
&= \tan A \tan B \tan C.
\end{aligned}$$

Now Cauchy's Inequality gives

$$\tan A + \tan B + \tan C \geq 3\sqrt[3]{\tan A \tan B \tan C}.$$

Thus

$$\tan A \tan B \tan C \geq 3\sqrt{3}.$$

By using Cauchy's Inequality again we obtain

$$\begin{aligned}
\tan^n A + \tan^n B + \tan^n C &\geq 3\sqrt[3]{\tan^n A \tan^n B \tan^n C} \\
&= 3\sqrt[3]{(\tan A \tan B \tan C)^n} \\
&\geq 3(\sqrt{3})^n \geq 3(1 + \frac{1}{2})^n \geq 3(1 + \frac{n}{2}) = 3 + \frac{3n}{2}.
\end{aligned}$$

Q1315 Let a and b be two integers. Prove that $2a + 3b$ is divisible by 17 if and only if $9a + 5b$ is divisible by 17.

ANS: First note that

$$3(9a + 5b) - 5(2a + 3b) = 17a.$$

Now if $2a + 3b$ is divisible by 17 then $5(2a + 3b)$ is a multiple of 17, and so is $3(9a + 5b)$. Since 3 is a prime, it follows that $9a + 5b$ is a multiple of 17. Conversely, if $9a + 5b$ is a multiple of 17, then so is $2a + 3b$ (because 5 is a prime).

Q1316 The area S and the angle γ opposite side c of a triangle are given. Determine the other two sides a and b such that c is minimum.

ANS: (Correct solution by J.C. Barton, Victoria)

The law of cosines gives

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos\gamma \\ &= (a-b)^2 + 2ab(1-\cos\gamma). \end{aligned}$$

On the other hand

$$S = \frac{1}{2}ab\sin\gamma$$

or

$$2ab = \frac{4S}{\sin\gamma}.$$

Therefore,

$$\begin{aligned} c^2 &= (a-b)^2 + 4S\frac{1-\cos\gamma}{\sin\gamma} \\ &= (a-b)^2 + 4S\frac{2\sin^2(\gamma/2)}{2\sin(\gamma/2)\cos(\gamma/2)} \\ &= (a-b)^2 + 4S\tan\frac{\gamma}{2}. \end{aligned}$$

Since S and γ are fixed, c^2 (and hence c) is minimum when $a = b$, i.e. the triangle is isosceles. In this case,

$$2a^2 = 2ab = \frac{4S}{\sin\gamma},$$

i.e.

$$a = b = \sqrt{\frac{2S}{\sin\gamma}}.$$

Q1317 Given a_1, a_2, a_3, a_4, a_5 satisfying

$$a_1^2 + \cdots + a_5^2 = 1,$$

prove that

$$\min_{1\le i\ne j\le 5}(a_i - a_j)^2 \le \frac{1}{10}.$$

ANS: Without loss of generality we can assume that

$$a_1 \le a_2 \le \cdots \le a_5.$$

Let $x \ge 0$ be such that

$$x^2 = \min_{1\le i\ne j\le 5}(a_i - a_j)^2.$$

Then for $i = 1, 2, 3, 4$ there holds

$$a_{i+1} - a_i \ge x,$$

so that for $j > i$

$$a_j - a_i = (a_j - a_{j-1}) + (a_{j-1} - a_{j-2}) + \cdots + (a_{i+1} - a_i) \geq (j-1)x.$$

Hence

$$(a_j - a_i)^2 \geq (j-i)^2 x^2.$$

By summing over i and j with $1 \leq i < j \leq 5$ we obtain

$$\sum_{j=2}^{5} \sum_{i=1}^{j-1} x^2 (j-i)^2 \leq \sum_{j=2}^{5} \sum_{i=1}^{j-1} (a_j - a_i)^2,$$

or

$$50x^2 \leq 5 \sum_{i=1}^{5} a_i^2 - \left(\sum_{i=1}^{5} a_i \right)^2 = 5 - \left(\sum_{i=1}^{5} a_i \right)^2 \leq 5.$$

Therefore, $x^2 \leq 1/10$.

Q1318 Find all primes a, b, and c satisfying

$$abc < ab + bc + ca.$$

ANS: Without loss of generality we can assume that $a \leq b \leq c$. Then

$$ab + bc + ca \leq 3bc.$$

If $a \geq 3$ then $3bc \leq abc$ so that $ab + bc + ca \leq abc$, which contradicts the given condition. Hence $a = 2$ (a is a prime). It follows that

$$2bc < 2b + bc + 2c,$$

or

$$\frac{1}{2} < \frac{1}{b} + \frac{1}{c}.$$

This implies that $b < 5$. So $b = 2$ or $b = 3$. If $b = 2$ then c can be any prime. If $b = 3$ then $c = 3$ or $c = 5$.

Q1319 Find all functions f satisfying

$$f\left(\frac{x+y}{2}\right) = \frac{f(x) + f(y)}{2}.$$

ANS: The question should have been "Find all *continuous* functions f satisfying ...". Let $a = f(0)$ and $b = f(1)$. Then

$$f(1) = \frac{f(0) + f(2)}{2} \quad \text{implying} \quad f(2) = 2(b-a) + a.$$

Similarly,

$$f(2) = \frac{f(1) + f(3)}{2} \quad \text{implying} \quad f(3) = 3(b - a) + a.$$

By induction,

$$f(m) = m(b - a) + a.$$

Next,

$$f(\frac{m}{2}) = \frac{f(m) + f(0)}{2} = \frac{m}{2}(b - a) + a.$$

By induction

$$f(\frac{m}{2^n}) = \frac{m}{2^n}(b - a) + a.$$

Now for any $x \in \mathbb{R}$ there exists a sequence x_1, x_2, x_3, ... of the form $m/2^n$ which converges to x (see note below). By the continuity of f we deduce

$$f(x) = (b - a)x + a.$$

Since a and b are two arbitrary numbers, f has the form

$$f(x) = cx + a \quad \text{for two real numbers } a \text{ and } c.$$

Note: To construct the sequence x_1, x_2, x_3, ... of the form $m/2^n$ converging to x we consider the case $0 < x < 1$. We define

$$\begin{aligned}
x_1 &= \frac{1}{2^{k_1}} && \text{where } k_1 \text{ is an integer satisfying } \frac{1}{2^{k_1}} < x \le \frac{1}{2^{k_1 - 1}} \\
x_2 &= x_1 + \frac{1}{2^{k_2}} && \text{where } k_2 \text{ is an integer satisfying } \frac{1}{2^{k_2}} < x - x_1 \le \frac{1}{2^{k_2 - 1}} \\
x_3 &= x_2 + \frac{1}{2^{k_3}} && \text{where } k_3 \text{ is an integer satisfying } \frac{1}{2^{k_3}} < x - x_2 \le \frac{1}{2^{k_3 - 1}} \\
&\dots \\
x_\ell &= x_{\ell-1} + \frac{1}{2^{k_\ell}} && \text{where } k_\ell \text{ is an integer satisfying } \frac{1}{2^{k_\ell}} < x - x_{\ell-1} \le \frac{1}{2^{k_\ell - 1}}
\end{aligned}$$

Then

$$x_\ell = \frac{1}{2^{k_1}} + \frac{1}{2^{k_2}} + \cdots + \frac{1}{2^{k_\ell}} = \frac{m}{2^n},$$

with $n = k_\ell$ and some integer m. Moreover,

$$0 < x - x_\ell = x - x_{\ell-1} - \frac{1}{2^{k_\ell}} < \frac{1}{2^{k_\ell - 1}} - \frac{1}{2^{k_\ell}} = \frac{1}{2^{k_\ell}}.$$

Since

$$k_1 < k_2 < k_3 < \cdots < k_\ell \to \infty \quad \text{as} \quad \ell \to \infty,$$

there holds

$$x_\ell \to \infty \quad \text{as} \quad \ell \to \infty.$$

The more general case of x is left as an exercise.

Q1320 Find all functions f satisfying

$$f\left(\frac{x+y}{2}\right) = \frac{2f(x)f(y)}{f(x)+f(y)}.$$

ANS: Similarly to the above question, this question should have been "Find all *continuous* functions f satisfying ...". If there is an x_0 such that $f(x_0) = 0$ then for any y

$$f(\frac{x_0+y}{2}) = \frac{2f(x_0)f(y)}{f(x_0)+f(y)} = 0,$$

so that $f(x) = 0$ for all x. But this function does not satisfy the given condition. Hence $f(x) \neq 0$ for all x. Let $g(x) = 1/f(x)$. Then g satisfies the condition of Q1319 so that $g(x) = cx + a$. Hence $f(x) = 1/(cx + a)$. In order that f is continuous, c must be 0. Hence f is a constant function.