Parabola Volume 46, Issue 3 (2010)

Factoring the "Unfactorable"

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I'm no math genius. Mr Boxold always said math is about thinking up new ways of looking at things, even old problems. That's all I did. I mean really, how do you expect a 16 year old, junior in high school, on a little island in the middle of the Pacific Ocean, to figure out something this unusual?

It's not like this hasn't been found before. But I can't count how many times I've heard or read: "You can't factor the sum of two squares over the reals." It's repeated all over the internet. There has to be an old time proof or demonstration somewhere. I mean it's not a huge secret.

Being the one to finally publish a proof that $a^2 = b^2$ can be factored over the reals is supposedly a huge honor. I mean it's not like I did anything my brother in Pre-Algebra couldn't do, OK never mind that, but you get the idea. It's real simple. I just found that by using your everyday factoring patterns you could factor $a^2 = b^2$ with real numbers.

How'd I do it? I walked into Pre-Calc one day in April and asked Mr Boxold what he was doing (pestering him is one of my favourite hobbies). Anyway he mumbled something about solving a Four Step Problem and continued working.

I watched him scratch out a few different attempts at the same problem before he figured it out. When he finished, it was completely different from the other attempts.

"Whoa, whoa, whoa, Mr. Boxold what the heck is that?" "I figured the problem Karina...duh." "Yeah but that's not how it's supposed to look!" "Math is about thinking up new ways to do old problems. I solved the problem." "Soooo you used the factoring pattern for $a^3 - b^3$ to solve for the Derivative of the Cube Root of x using the Four Step Rule?" "Yeah"

The next class I was looking at the problem he had solved...

He had used $a^3 - b^3$ factoring pattern but he used $x^{1/3}$ instead. I had an idea...

> I knew the factoring pattern for the Sum of Cubes... If it would work for Cube Roots... Then maybe I could write the powers as $6/3$ and factor the Sum of Squares...

I played around with it and... (dramatic pause) it worked! I was pretty surprised, and showed Mr. Boxold. He was impressed and said it showed that while the Sum of Squares *can* be factored over the reals, the only *linear* factorization is over the imaginaries. Then he told me there are ways to factor the sum of any odd power. He explained that the rule for the Sum of Quintics had alternating signs and suggested I figure out

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the rest. He couldn't see why they couldn't all be used to factor the Sum of Squares. I was interested and had nothing else to do so I tried it out. This demonstrated that not only is the Sum of Squares factorable but there are also infinity ways of factoring it. Mr. Boxold told me that I really needed to work out the general formula and show that it always collapsed to the Sum of Squares. So I wrote the formula that can be used to factor the Sum of Squares over ANY odd power and got this:

$$\begin{aligned}
a+b &= (a^{1/3}+b^{1/3})(a^{2/3}-a^{1/3}b^{1/3}+b^{2/3}) \\
a^3+b^3 &= (a+b)(a^2-ab+b^2) \\
a^5+b^5 &= (a+b)(a^4-a^3b+a^2b^2-ab^3+b^4)
\end{aligned}$$

For n odd

$$a^n+b^n=(a+b)(a^{n-1}-a^{n-2}b+\cdots-ab^{n-2}+b^{n-1})$$

We also have

$$\begin{aligned}
&a^2+b^2=a^{6/3}+b^{6/3} \\
&\quad=(a^{2/3}+b^{2/3}) \\
&\quad\times(a^{4/3}-a^{2/3}b^{2/3}+b^{4/3})
\end{aligned}$$

$$\begin{aligned}
&a^2+b^2=a^{10/5}+b^{10/5} \\
&\quad=(a^{2/5}+b^{2/5}) \\
&\quad\times(a^{8/5}-a^{6/5}b^{2/5}+a^{4/5}b^{4/5}-a^{2/5}b^{6/5}+b^{8/5})
\end{aligned}$$

$$\begin{aligned}
&a^2+b^2=a^{14/7}+b^{14/7} \\
&\quad=(a^{2/7}+b^{2/7}) \\
&\quad\times(a^{12/7}-a^{10/7}b^{2/7}+a^{8/7}b^{4/7}-a^{6/7}b^{6/7}+a^{4/7}b^{8/7}-a^{2/7}b^{10/7}+b^{12/7})
\end{aligned}$$

In general we have:

$$\begin{aligned}
&a^2+b^2=a^{2n/n}+b^{2n/n} \\
&\quad=(a^{2/n}+b^{2/n}) \\
&\quad\times(a^{(2n-2)/n}-a^{(2n-4/n)}b^{2/n}+a^{(2n-6)/n}b^{4/n}- \\
&\quad\cdots+a^{4/n}b^{(2n-6)/n}-a^{2/n}b^{(2n-4)/n}+b^{(2n-2)/n})
\end{aligned}$$

To check this, when we take the product and distribute $a^{2/n}$ from the left, we get the series:

$$\begin{aligned}
&a^{2n/n}-a^{(2n-2)/n}b^{2/n}+a^{(2n-4)/n}b^{4/n}- \\
&\quad\cdots+a^{6/n}b^{(2n-6)/n}-a^{4/n}b^{(2n-4)/n}+a^{2/n}b^{(2n-2)/n}
\end{aligned}$$

When we take the same product and distribute b^2/n from the right, we get:

$$\begin{aligned}
&a^{(2n-2)/n}b^{2/n}-a^{(2n-4)/n}b^{4/n}+a^{(2n-6)/n}b^{6/n}- \\
&\quad\cdots+a^{4/n}b^{(2n-4)/n}-a^{2/n}b^{(2n-2)/n}+b^{2n/n}
\end{aligned}$$

Adding these two series together causes all the terms to cancel except $a^{2n/n}$ and $b^{2n/n}$. This leaves us with $a^2 + b^2$. Thus there are an infinity of ways to factor the Sum of Squares over the reals.