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Solutions to Problems 1331-1340

Q1331 Given any positive integers m and n prove that every divisor of mn can be expressed as a product of a divisor of m and a divisor of n.
ANS: Let d be a divisor of mn, which we denote by $d|mn$, and let d_1 be the greatest common divisor of d and m, which we write by $d_1 = \text{g.c.d.}(d, m)$. We can now write

$$\begin{aligned} mn &= kd \\ d &= d_1 d_2 \\ m &= d_1 m_1 \end{aligned}$$

with $k, d_2, m_1 \in \mathbf{Z}^+$ and g.c.d$(m_1, d_2) = 1$. It now follows that $kd_1d_2 = m_1d_1n$ so that $kd_2 = m_1n$ and as g.c.d.$(m_1, d_2) = 1$ then $d_2|n$ and thus $d = d_1d_2$ with $d_1|m$ and $d_2|n$.

Q1332 Show that 7999999999999999 is not a prime number.
ANS:

$$\begin{aligned} 7999999999999999 &= 200000^3 - 1^3 \\ &= (200000 - 1)(200000^2 + 200000 \times 1 + 1^2) \\ &= 199999 \times 40000200001. \end{aligned}$$

Q1333 Find the largest coefficient when $(x + 2y + 3z)^{99}$ is expanded and like terms collected.
ANS: As there are only finitely many coefficients the maximum clearly exists. Suppose that the maximum is the coefficient of $x^py^qz^r$, which is $a_{p,q,r} = (99!/p!q!r!)2^q3^r$. It is not too hard to show that the cases $p, q, r = 0, 99$ do not give a maximum. So we have $a_{p,q,r} \geq a_{p,q-1,r+1}$ and $a_{p,q,r} \geq a_{p-1,q,r+1}$ which leads to $2(r+1) \geq 3q$ and $r+1 \geq 3p$. Since $p + q + r = 99$ we have $297 = 3p + 3q + 3r \leq 6r + 3$ and hence $r \geq 49$. Similarly we obtain $q \geq 33$ and $p \geq 16$. Using again the fact that $p + q + r = 99$, we have only three possible cases, $p = 17, q = 33, r = 49$ or $p = 16, q = 34, r = 49$ or $p = 16, q = 33, r = 50$. It is not hard to show that $a_{16,34,49} = a_{17,33,49} = (50/51)a_{16,33,50}$ and so the maximum is $a_{16,33,50} = (99!/16!33!50!)2^33^50$.

Q1334 Suppose you have 100 lightbulbs numbered from 1 to 100, and that each lightbulb has a push button on/off switch. Initially, all lightbulbs are off. Now consider the following sequence of steps: In the first step, press the switches for those lightbulbs whose numbers are divisible by 1 (i.e. turn on all lightbulbs). In the second step, press the switches for those lightbulbs whose number is divisible by 2. Continue in this fashion and in the k-th step, press the switches for those lightbulbs whose number is divisible by k.

After 100 steps, which lightbulbs are switched on?
ANS: First note that the lightbulbs that are on after the given sequence of steps are those that have been switched an odd number of times. Thus the problem can be rephrased as: "Find all numbers from 1 to 100 which have an odd number of divisors." Here we include one as a divisor and then this immediately excludes all primes p which have two divisors 1 and p. More generally the factors of a number come in pairs: if d is a factor of n then so is $\frac{n}{d}$ and so the total number of factors of n is even. The only exception to this is the case where d and $\frac{n}{d}$ are equal, that is $n = d^2$. The additional factor d in this case makes the total number of factors odd. Thus the lightbulbs which are on after 100 steps are exactly those whose number is a perfect square, that is,

$$1, 4, 16, 25, 36, 49, 64, 81, 100.$$

Q1335 What is the probability that two or more students in a class of thirty share the same birthday?
ANS: First we make two assumptions: (i) there are 365 days in a year. (ii) birthdays are uniformly distributed throughout the year, so that a person has a $\frac{1}{365}$ chance of having their birthday on any specified date. The probability that all thirty students have birthdays on different days is

$$p = \frac{364}{365} \times \frac{363}{365} \times \frac{362}{365} \times \dots \frac{365 - 29}{365}.$$

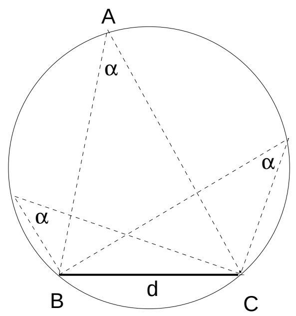
Thus the probability that two or more of the students share the same birthday is $1 - p$ which evaluates to approximately 70%.

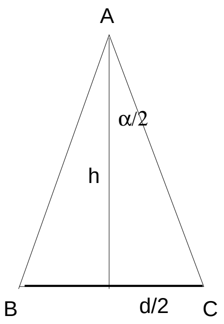
Q1336 Two lighthouses stand a distance d km apart along the coast. A ship sails from one to the other, taking a course such that the angle between the lighthouses, as seen from the ship, is always α. What is the area of the largest triangle formed by the ship and the two lighthouses in the course of the voyage?

ANS: This is the problem of finding the area of the triangle ABC with the largest area given that the length of one side $d = BC$ and the opposite angle α are fixed. Representative triangles are shown in the figure below left. Basic geometry shows that the path of the ship is a circle. As the triangle has a fixed base, the one with the largest area also has the largest altitude, which is the isosceles triangle shown below right with area

$$A = 2\left(\frac{d}{2}\right) h = \frac{d^2}{4} \cot(\frac{\alpha}{2}).$$

Q1337 On an $n \times n$ chessboard ($n \geq 2$) we wish to colour a 2×2 block of squares red and another 2×2 block blue. The red and blue blocks may not overlap. In how many ways can this be done?
ANS: The red block can be placed in $(n-1)^2$ ways; allowing for overlap, the blue block





can be placed in the same number of ways. Now we have to eliminate overlaps. Blocks overlap at a corner square if and only if their convex hull is a 3×3 square; there are $(n-2)^2$ ways to place this, and 4 ways to place the red and blue blocks within the 3×3. Blocks overlap "along a side" if and only if their convex hull is a 2×3 or 3×2 rectangle; there are $(n-1)(n-2)$ possibilities in each case, and then two possibilities for placing the red and blue blocks. Finally, the red and blue blocks could coincide. So the number of admissible placements is

$$(n-1)^4 - 4(n-2)^2 - 4(n-1)(n-2) - (n-1)^2 = (n-1)^4 - (3n-5)^2 \,.$$

`Alternative argument`. The red block can be in any one of $(n-1)^2$ places. The top - left corner of the blue block can then go anywhere except (i) along the right or bottom sides of the board, as then the blue block extends beyond the board; (ii) inside, or immediately to the left of or above the red block, as then the blocks overlap. Out of the $(n-1)^2$ positions initially available to the blue block, (i) 4 are ruled out if the red block is in a corner of its "possible square"; (ii) 6 are ruled out if the red block is on a side of its possible square; (iii) 9 are ruled out if the red block is within its possible square. So the number of possibilities to be removed is

$$9(n-3)^2 + 6 \times 4(n-3) + 16 = (3n-5)^2$$

as above.

Q1338 Given any two odd integers, a and b, prove that $a-b$ is divisible by 2^{2010} if $a^{2011} - b^{2011}$ is divisible by 2^{2010}.
ANS: The basic factorization formula gives

$$a^{2011} - b^{2011} = (a-b)(a^{2010} + a^{2009}b +^{\cdots} b^{2009}a + b^{2010})$$

and the factor

$$(a^{2010} + a^{2009}b +^{\cdots} b^{2009}a + b^{2010})$$

must be odd because it has an odd number (2011) of terms, each of which is odd. It follows that if $a^{2011} - b^{2011}$ is divisible by 2^{2010} then $(a-b)$ is also divisible by 2^{2010}.

Q1339 The cubic equation $x^3 - 2x^2 - 3x - 4$ has three solutions a, b, c, all different. Prove that

$$\frac{a^{2010} - b^{2010}}{a - b} + \frac{b^{2010} - c^{2010}}{b - c} + \frac{c^{2010} - a^{2010}}{c - a}$$

is an integer.
ANS: If

$$S_n = \frac{a^n - b^n}{a - b} + \frac{b^n - c^n}{b - c} + \frac{c^n - a^n}{c - a}$$

then

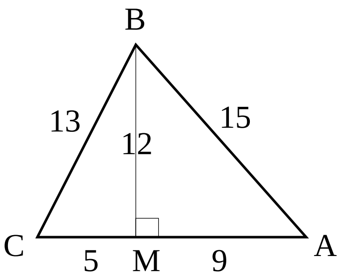
$$S_0 = 0 \,, \quad S_1 = 3 \quad \text{and} \quad S_2 = 2(a + b + c) = 4$$

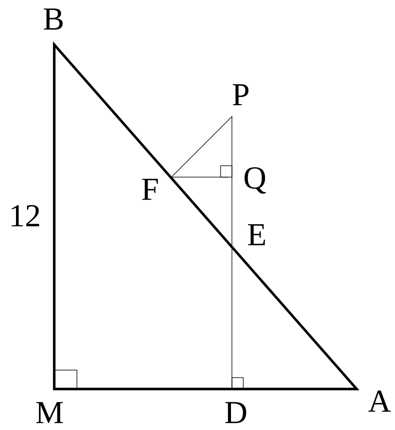
are integers, and

$$S_{n+3} = 2S_{n+2} + 3S_{n+1} + 4S_n \,.$$

Hence S_n is an integer for all n.

Q1340 ΔABC has side lengths $AB = 15$ cm, $AC = 14$ cm, $BC = 13$ cm. Its area is 84 cm^2. The point P lies closest to the side AB but outside the triangle. The distances from P to AB and AC are 3 cm and 10 cm respectively. Find the distance of P to BC.
ANS: Given the area is 84, the altitude from the base of length 14 is 12. The triangle ABC is shown on the left in the figure following.





Now consider the triangle shown on the right with exterior point P closest to the side AB. We have also marked points D, E, F, Q to identify similar triangles

$$\triangle FPQ \sim \triangle EPF \sim \triangle EAD \sim \triangle BAM.$$

It follows that

$$PQ = \frac{9}{5},\ FQ = \frac{12}{5},\ FE = 4,\ PE = 5.$$

But $PD = 8$ so that

$$ED = 4,\ AE = \frac{15}{4},\ AD = \frac{9}{4}.$$

So if C is the origin then $P = (\frac{47}{4}, 8)$ and the distance from the line BC is

$$\frac{(12.\frac{47}{4} - 5.8)}{13} = \frac{101}{13}.$$