

Solutions 1341–1350

Q1341 A lazy weather forecaster predicts that future maximum temperatures will be the average of the preceding two days maximum temperatures. The forecaster starts his forecast by noting that yesterday's maximum temperature was 23°C and the day before it was 29°C . What temperature in $^{\circ}\text{C}$ is the weather forecaster's long term maximum temperature forecast?

ANS: The forecast temperature $T(n)$ on day n satisfies the recurrence relation

$$T(n) = \frac{1}{2}T(n-1) + \frac{1}{2}T(n-2)$$

with initial conditions $T(1) = 29$, $T(2) = 23$.

If we seek a trial solution of the recurrence relation in the form, $T(n) = a\lambda^n$ then we find this is a valid solution for any a provided that λ is a solution of the quadratic equation $\lambda^2 - \frac{1}{2}\lambda - \frac{1}{2} = 0$. The solution of the quadratic equation yields $\lambda = 1$ and $\lambda = -\frac{1}{2}$ thus

$$T(n) = a + b\left(-\frac{1}{2}\right)^n.$$

If we use the initial conditions $T(1) = 29$, $T(2) = 23$ then we find that a and b must satisfy the simultaneous equations

$$\begin{aligned}a - \frac{b}{2} &= 29 \\a + \frac{b}{4} &= 23.\end{aligned}$$

We then solve for $a = 25$ and $b = -8$ so that $T(n) = 25 - 8\left(-\frac{1}{2}\right)^n$ and for large n we find the long term forecast $T = 25^{\circ}\text{C}$.

Q1342 An employer is negotiating with a union on future staff salary increases. The union asks for an outcome with a flat salary increase of 18% at the end of 4.5 years. The employer offers two 2% pay rises each year with the first increase in January and the second in July. The Treasury estimates the CPI growth to be 3.5% per year for the next 4.5 years.

Does the employer's offer match the union request? Does the employer's offer match the CPI?

ANS: Suppose that the staff member starts off with a salary of S . Their salary after nine increases, each at 2% of their current salary, will be

$$S = S\left(1 + \frac{2}{100}\right)^9 \approx S\left(1 + \frac{19.5}{100}\right).$$

Thus the salary will be 19.5% higher at the end of the period. This could be called a flat rate of 18% by ignoring the compounding effects since $9 \times \frac{2}{100} = \frac{18}{100}$. So the employer's offer matches the union request.

Staff are generally more concerned about how much money they receive to pay bills rather than the salary for their sector. The two are not unrelated but a staff member starting the year with a salary of S would receive an income of

$$I = \frac{1}{2}S(1 + \frac{2}{100}) + \frac{1}{2}S(1 + \frac{2}{100})^2 = S(1 + \frac{3.02}{100})$$

over the course of the year. This represents a growth in income of 3.02% per annum, quite a bit less than the growth in the salary rate. This may or may not match a growth in the CPI of 3.5% over a year. It depends on the time course of the growth of a typical "basket of goods".

Q1343 Imagine that you stand at the origin of a number line and take a random walk along the line directed by the toss of a coin. If the coin shows heads you take a step to the integer on the left and if the coin shows tails you take a step to the integer on the right. After $2n$ tosses of the coin what is the probability that you will be at position n ?

ANS: First notice that after an even number of tosses you must have taken an even number of left steps and an even number of right steps, or an odd number of each; in either case you must end up at an even position. So if n is odd the probability is 0. If n is even, let $n = 2m$ and suppose that you have taken k steps to the right. Then you have taken $4m - k$ steps to the left and your final position is $k - (4m - k) = 2k - 4m$, which equals $2m$. Hence $k = 3m$. The number of possible results from $4m$ tosses of a coin is 2^{4m} , and the number in which $3m$ are heads is

$$C(4m, 3m) = \frac{(4m)!}{(3m)! m!}.$$

Therefore the required probability is

$$\frac{(4m)!}{2^{4m} (3m)! m!}$$

if $n = 2m$, and zero if n is odd.

Q1344 A man is thirty years old. His sister is twice as old as he was when she was thirty years old. How old is the woman?

ANS: Let the woman's age now be w years. Then the difference between her age and her brother's is $w - 30$. When she was 30 the brother was $\frac{1}{2}w$ and so the difference was $30 - \frac{1}{2}w$. But the difference between two people's ages never changes. Therefore

$$w - 30 = 30 - \frac{1}{2}w$$

and we easily solve to get $w = 40$. So the woman is 40 years old.

Q1345 Note that the function $g(x) + \frac{1}{1-x}$ has the properties

$$g(x) \neq x, \quad g(g(x)) \neq x, \quad g(g(g(x))) = x.$$

In a simpler notation we can write this as

$$g(x) \neq x, \quad g_2(x) \neq x, \quad g_3(x) = x.$$

Find real numbers p, q, r, s such that the function

$$h(x) = \frac{px + q}{rx + s} \tag{1}$$

has the properties

$$\begin{aligned} h(x) \neq x, \quad h_2(x) \neq x, \quad h_3(x) \neq x, \\ h_4(x) \neq x, \quad h_5(x) \neq x, \quad h_6(x) = x. \end{aligned}$$

ANS: In view of what we know about $g(x)$, we begin by trying to find p, q, r, s such that

$$h(h(x)) = g(x),$$

that is,

$$\left(p \frac{px + q}{rx + s} + q\right) / \left(r \frac{px + q}{rx + s} + s\right) = \frac{1}{1-x}.$$

Simplifying the left-hand side gives

$$\frac{(p^2 + qr)x + (pq + qs)}{(pr + rs)x + (qr + s^2)} = \frac{1}{1-x}; \tag{2}$$

then multiplying out and equating coefficients leads to the simultaneous equations

$$p^2 + qr = 0, \quad -(pq + qs) = pr + rs, \quad pq + qs = qr + s^2.$$

The second equation can be rewritten and factorised as

$$(p + s)(q + r) = 0$$

and so we have two possibilities.

1. If $p = -s$ then $qr + s^2 = 0$; this is impossible since it means that (2) has a zero denominator.
2. If $r = -q$ then $p^2 - q^2 = 0$ and $s^2 - qs - q^2 - pq = 0$, so we can find p and s in terms of q : a little bit of algebra gives four possibilities

$$\begin{aligned} p = q, \quad r = -q, \quad s = -q; \\ p = q, \quad r = -q, \quad s = 2q; \\ p = -q, \quad r = -q, \quad s = 0; \\ p = -q, \quad r = -q, \quad s = q. \end{aligned}$$

Now we already know that the first and fourth are impossible; the third gives

$$h(x) = \frac{-qx + q}{-qx} = \frac{x - 1}{x} ,$$

and doing the algebra gives

$$h(h(h(x))) = x$$

which is not what we want. Finally, the second possibility gives

$$h(x) = \frac{qx + q}{-qx + 2q} = \frac{1 + x}{2 - x} ,$$

and checking the necessary algebra shows that this works.

Q1346 (a) Find a function $f(x)$ defined for all real numbers $x \neq 0, 1$ such that the equation

$$f(x) + x^2 f\left(\frac{1}{1-x}\right) = x$$

is true for all x .

(b) Prove that the equation

$$f(x) + x^3 f\left(\frac{1}{1-x}\right) = x$$

has no solution.

ANS: To handle both problems simultaneously, consider the equation

$$f(x) + a(x)f(g(x)) = b(x) , \tag{1}$$

where

$$g(x) = \frac{1}{1-x} .$$

Replacing x by $g(x)$ in (1) gives

$$f(g(x)) + a(g(x))f(g(g(x))) = b(g(x)) . \tag{2}$$

Doing the same thing in (2) yields

$$f(g(g(x))) + a(g(g(x)))f(g(g(g(x)))) = b(g(g(x))) . \tag{3}$$

Calculating the details, we have

$$g(g(x)) = \frac{1}{1-g(x)} = \frac{1-x}{(1-x)-1} = \frac{x-1}{x}$$

and, amazingly,

$$g(g(g(x))) = \frac{g(x)-1}{g(x)} = \frac{1-(1-x)}{1} = x .$$

Therefore (3) can be written as

$$f(g(g(x))) + a(g(g(x)))f(x) = b(g(g(x))) , \quad (4)$$

and we can now treat (1), (2) and (4) as three equations in the three “unknowns” $f(x)$, $f(g(x))$ and $f(g(g(x)))$. The easiest way to solve them is to take equation (1), subtract $a(x)$ times equation (2), and add $a(x)a(g(x))$ times equation (4): we find

$$\begin{aligned} (1 + a(x)a(g(x))a(g(g(x))))f(x) \\ = b(x) - a(x)b(g(x)) + a(x)a(g(x))b(g(g(x))) . \end{aligned}$$

For the first question we substitute $a(x) = x^2$ and $b(x) = x$. This gives

$$\begin{aligned} \left(1 + x^2 \left(\frac{1}{1-x}\right)^2 \left(\frac{x-1}{x}\right)^2\right)f(x) \\ = x - x^2 \frac{1}{1-x} + x^2 \left(\frac{1}{1-x}\right)^2 \frac{x-1}{x} , \end{aligned}$$

which simplifies to

$$2f(x) = \frac{2x^2}{x-1} .$$

So

$$f(x) = \frac{x^2}{x-1}$$

is a potential solution of (a), and calculating the left-hand side confirms that it actually is a solution. For (b), we take $a(x) = x^3$ and $b(x) = x$. After simplification we obtain the equation

$$0 f(x) = \frac{x - 3x^2 + x^4}{(1-x)^2} ,$$

and as this is supposed to be true for all x there is no solution.

Q1347 On an 8×8 chessboard, how many ways are there to paint five of the squares red, (a) if there must be a horizontal row of at least four consecutive red squares? (b) if there must be a row (in any direction) of at least four consecutive red squares?

ANS: In any row there are five possible choices of four consecutive squares. So to make a choice of five squares to be painted red in part (a) we may do the following.

1. Choose four consecutive squares
in a horizontal row 40 ways
2. Choose any fifth square 60 ways

This gives 2400 possibilities. However, consider arrangements in which all five red squares occur consecutively in a horizontal row. There are $4 \times 8 = 32$ such arrangements, and in the preceding scheme each one has been counted twice. So we must deduct 32 to get the correct answer of 2368.

Similar ideas will count the possibilities for a diagonal row of four red squares in, let's

say, the “northeast to southwest” direction. There are diagonal lines of one, two or three squares which obviously cannot contain four consecutive red squares; a line of four which can; a line of five which can do so in two ways; and so on up to a line of eight which can do so in five ways. Then there are further diagonals of seven, six, five and four squares. Choosing one of these possibilities and then any fifth square gives a preliminary answer of

$$(1 + 2 + 3 + 4 + 5 + 4 + 3 + 2 + 1) \times 60 = 1500 .$$

By a similar argument, the number of ways to choose five consecutive square on a “NE–SW” diagonal is

$$1 + 2 + 3 + 4 + 3 + 2 + 1 = 16 ,$$

and for the same reasons as in (a), this must be subtracted from the total to give 1484. Finally, we must find the number of arrangements with four red squares in a vertical line or in a “NW–SE” diagonal; but these are the same as the two cases we have already considered. So our final answer for (b) is

$$2 \times (2368 + 1484) = 7704 .$$

Q1348 Let p and q be consecutive primes other than 2. Show that $p + q$ is a product of at least three prime numbers (not necessarily all different).

ANS: We have

$$p + q = 2 \left(\frac{p + q}{2} \right)$$

and $p + q$ must be even so the second factor must be an integer, which is also greater than p but less than q . But p and q are *consecutive* primes so that the second factor is not prime, i.e., the second number is a composite so that $p + q$ is composed of at least three factors.

Comment. If 2 was not excluded then p and q could be 2 and 3; then $p + q$ does not have three prime factors.

Correct solution received from John Barton, Victoria.

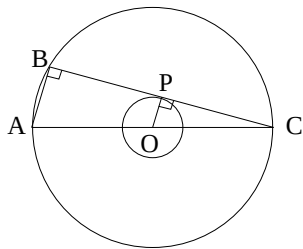
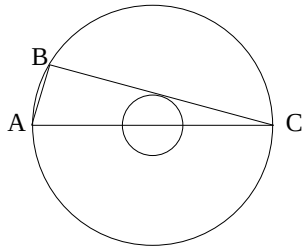
Q1349 In the following figure the larger circle has a radius four times larger than the radius of the smaller circle. AC passes through the centre of the concentric circles. What is the length of the chord AB if the radius of the larger circle is 24?

ANS. Construct the radius OP from the centre of the smaller circle to the point of tangency along BC . Note that $\angle ABC$ and $\angle OPC$ are both right angles and $\triangle ABC \sim \triangle OPC$. We also have $OC = 4OP$. The length of the chord AB is found from

$$\frac{AB}{OP} = \frac{AC}{OC} = 2$$

and $OP = \frac{OC}{4} = 6$ hence $AB = 12$.

Correct solution received from John Barton, Victoria.



Q1350 Alan, Betty, Chris and Debbie are of different ages. They all speak the truth whenever they talk to an older person, but they all lie when talking to a younger person. Alan says to Betty, "I am older than Chris". Debbie says to Chris, "You are the youngest of us all." List the four people in order from youngest to oldest.

ANS: If Alan's statement is true then Alan is younger than Betty (because he has told the truth) and older than Chris (because the statement is true). If Alan's statement is false then Alan is older than Betty and younger than Chris. So these three must be in the order Chris, Alan, Betty or Betty, Alan, Chris. Now Debbie's statement cannot be true as she would then have told the truth to a younger person. So it is false, which means that Debbie is older than Chris and Chris is not the youngest of the four. This shows that the first of our earlier options must be ruled out, and the order of ages must be Betty, Alan, Chris, Debbie.

Correct solution received from John Barton, Victoria.