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Subliminal Deduction?

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Let me begin by recounting a story I first heard almost fifty years ago. This is, of course, a long time and over the intervening years I have lost contact with the people involved. So it may well be that I have misremembered parts of it and the actual event may well have been somewhat different in its details. However, I will tell it as I recall it, and its general gist is enough to introduce my topic for this column.

Two graduate students, contemporaries of mine, married, saw the birth of a son and went together to study abroad. On one occasion, the wife went to the dentist, leaving the youngster in the care of his father, a somewhat absent-minded man. Sitting in the dentist's chair, the wife suddenly had an overpowering feeling that something awful had happened to the boy. So strong was this feeling that she got up at once and rushed home, to find that he had indeed fallen down a flight of stairs and suffered concussion, without the father's being aware of the event. Fortunately his fall had been broken to some extent and his condition was not after all very serious, but this happy circumstance does not alter the remarkable impact of the story itself. I learned the tale from the wife at a social gathering that included the husband and some others, one of whom saw it as a clear example of ESP. However, the husband in particular, seemed reluctant to accept this idea, as indeed was I. One does not readily take to such an explanation, but if we reject it, then we are left with the task of producing another.

Here is mine.

The wife may well have recalled that the flight of stairs was accessible to the child because a door had been left open, and knowing the toddler's inquisitive nature and her husband's absent-mindedness, she may well have been led to imagine such a catastrophe. As it turned out, both her memory and her appraisals of the child's and the father's natures were accurate. She *deduced* the likely course of events, and her deduction proved correct.

In such a case, the *route* by which she reached her conclusion is much less important than the conclusion itself. When she told the story, she could not say how it was that she arrived at it. If my explanation is correct, then the chain of deduction that led to her concern was either entirely subconscious or else subsequently forgotten in the ensuing confusion. I am calling such deductions *subliminal* in analogy with the term "subliminal perception" in which we perceive something without being consciously aware of it. The existence of subliminal perception is well established and has even found application in the advertising industry! Here I am positing that we may also *deduce* matters without the intervention of our conscious mind.

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The existence of something like this is a key element in the thinking of the philosopher and scientist Michael Polanyi. In his book *Personal Knowledge* (first published 1958), he addresses the possibility that one may know something without being able to explain *how* we know it. He later summed it up as: "*we can know more than we can tell*". For example, when we learn to ride a bicycle, we acquire the skill and, in that sense, know how it is done, but most of us never know at all the complicated dynamics involved in staying upright. Nor need we; in fact if we tried to apply such knowledge in the practical situation, we would probably fall off! This is one of Polanyi's examples. He adduces many of them. On p. 88 of *Personal Knowledge*, he provides this summary:

> Although the expert diagnostician, taxonomist and cotton-classer can indicate their clues and formulate their maxims, they know many more things than they can tell, knowing them only in practice, as instrumental particulars, and not explicitly as objects ... . [Such things] can be taught only by aid of practical example and never solely by precept.

An instance of this concerns the art of chicken sexing. It is very difficult to assess the sex of a day old chick, but it can be very important to the poultry industry that this be done and be done correctly. There are current attempts to breed into poultry lines genetic markers that make the sex-difference readily observable. When such markers are not available, however, expert "chicken sexers" are employed; these people can produce very reliable results (with errors less than 0.5%) but nonetheless can often be quite unable to explain their methods.

I am not sure now where I first heard this example used to illustrate Polanyi's thought, but it has stuck with me and has also intrigued others. Thus, for example, we find J. H. Gill saying (in his book *The Tacit Mode: Michael Polanyi's Postmodern Philosophy* [SUNY Press, 2003], p. 43):

> As John Dewey said, we learn by doing. In attempting to practice an art or skill, we indwell it to the extent that it eventually comes to indwell us, even though we cannot say how this happens. In learning a new dance step, a new language, or how to think philosophically, there simply is no substitute for practice. We imitate, are corrected, try again and again, get corrected again, and gradually get better at the task. Perhaps the ultimate example of this process is exhibited by those folks who are trained to be 'chicken sexers'. Even though there is no simple way to tell the sex of a tiny chick, people can be taught to sort the males from the females by apprenticing themselves to those who already know how to do so. Their awareness becomes a function of their activity.

This sort of thing however, is the exact opposite of the situation that obtains in Mathematics (and indeed in scholarship more generally). It is not enough to say *what* it is we know, but we must also say *how* we know it. In Mathematics, this requirement is the insistence on *proof*. We do not claim a mathematical result to be true unless we can *prove* it.[2] Thus, until recently, "Fermat's Last Theorem" was actually not a theorem but a *conjecture*, one backed by a wealth of supporting evidence, but all the same lacking actual proof.

Polanyi's position is well explained in this quotation from the *infed* website: http://www.infed.org/thinkers/polanyi.htm

> Central to [his] thinking was the belief that creative acts (especially acts of discovery) are shot-through or charged with strong personal feelings and commitments (hence the title of his most famous work *Personal Knowledge*). Arguing against the then dominant position that science was somehow value-free, [he] sought to bring into creative tension a concern with reasoned and critical interrogation with other, more 'tacit', forms of knowing.
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> [His] argument was that the informed guesses, hunches and imaginings that are part of exploratory acts are motivated by what he describes as 'passions'. They might well be aimed at discovering 'truth', but they are not necessarily in a form that can be stated in propositional or formal terms. As [he] wrote in [a later book] *The Tacit Dimension*, we should start from the fact that 'we can know more than we can tell'. He termed this pre-logical phase of knowing as 'tacit' knowledge. Tacit knowledge comprises a range of conceptual and sensory information and images that can be brought to bear in an attempt to make sense of something ... Many bits of tacit knowledge can be brought together to help form a new model or theory. This inevitably led him to explore connoisseurship and the process of discovery (rather than with the validation or refutation of theories and models ...).

Another author who has explored this matter, although from a different viewpoint, is the mathematician Jacques Hadamard ($1865 - 1963$). Because his study is explicitly concerned with Mathematics, I shall concentrate on it rather than Polanyi's, although the two authors cover much of the same ground, and Polanyi actually makes reference to Hadamard. Hadamard was a most eminent mathematician. Among his many achievements is a theorem showing that the number of primes less than a given number n is approximately $n/\ln n$; there were also major contributions to other aspects of number theory and to analysis (advanced calculus). A class of matrices is named in his honor . A Jew, he fled the German occupation of his native France for the United States in 1940, and it was in the US that he wrote the work I discuss here. It is entitled *An Essay on the Psychology of Invention in the Mathematical Field*, and it is a remarkable work for a pure mathematician to have produced. Its inspiration came from a lecture by his friend and mentor Henri Poincaré ($1854 - 1912$), who outlined the remarkable chain of discovery of one of his deepest discoveries, relating how key ideas came into his mind

[2]Since Gödel's Theorem, we have come to accept that there may be statements that are "true but unprovable", but these are different from the theorems of mainstream Mathematics. If a mathematical conjecture is "true but unprovable", all we can say is that we will never know this to be so! We could only assert such a status on the strength of a proof or refutation; such would either make the conjecture *provable* or else *untrue*: a clear contradiction! The question of the connection between truth and proof in such contexts is a different matter from that under discussion here.

"out of the blue", when he was occupied with quite other matters. [The details are extremely technical and I will not attempt to give them here.]

Hadamard set out to explore the question of how such inspiration occurs, and his book is the result. His first discussion concerns his title. Is "invention" the right word? Should we perhaps say "discovery"? This is a somewhat vexed question; I devoted my column in *Parabola Incorporating Function*, February 2002 to a closely related matter. The issue boils down to: "Do mathematical results pre-exist our interest in them? – Are they, like America, already out there simply awaiting discovery?" Hadamard quotes another of his mentors, Charles Hermite ($1822 - 1901$) to the effect that new mathematical results are really discoveries of pre-existing verities. In fact, Hermite is on record as saying: "There exists, if I am not mistaken, an entire world which is the totality of mathematical truths, to which we have access only with our mind, just as the world of physical reality exists, the one like the other independent of ourselves, both of divine creation."

Nonetheless, Hadamard chooses to use the word "invention" (as opposed to "discovery") because the former emphasizes the important rôle of the mathematician, and suggests an activity of *creativity*. It is this aspect that informs much of his subsequent discussion.

When he considers Poincaré's testimony as to how he came upon his results, he places importance on the idea of putting a problem aside and thinking about something else. It is not uncommon in such cases for the key to the puzzle to pop into one's mind, quite uninvited, so to speak. We all have experience of such occurrences. Our minds must somehow still be working on the problem before us, although we are not consciously addressing it.

However, both Polanyi and Hadamard are insistent that their descriptions apply to the later aspects of the discovery or invention. Receiving a conviction "out of the blue" that a result holds cannot occur without preparatory conscious thought, nor does it absolve one from the hard grind of finding a valid proof. In the case of Poincaré, this was something he did "for conscience' sake", as he put it, but it is also clear that it involved a lot of quite difficult work. Without such a later step, there is always the possibility that the initial conviction was misplaced: that an error has been made.

All in all, Hadamard identifies four stages in the creative process. The first, *preparation*, is conscious; it is that aspect of the work that introduces the mind to the matter under investigation. The second, *incubation*, is unconscious; it takes place when one's mind is otherwise engaged, but (quite clearly) still working on the matter subconsciously, or as I would say, subliminally. The third stage is *illumination*, when the work done by the subconscious emerges into consciousness, what some psychologists refer to as "the *aha* phenomenon". Finally, there is the fourth stage, which is conscious. This is the process of verification, followed perhaps by a reordering of the proof to make it easier to communicate.

This description certainly covers the case in a lot of mathematical research. It is probably not a universal paradigm; there is really no reason why stage 2 *has* to be subconscious, although much of the time it surely is. However, one may well embark on a complex line of reasoning not knowing where it will lead. Consider in this regard the case of a detailed computation. It may be that such cases are more routine than the flashes of inspiration that inform the bulk of Hadamard's text, but I would hesitate to say that such activities were not creative. But what is certain is that he has described and categorized much of the process of creativity and has drawn particular attention to the rôle of the subconscious.

He deals also with four cases in which the final stage is absent. He calls these "intuitions". His first example is that of "Fermat's Last Theorem". The story has been told many times, but it bears repeating here. Fermat possessed a copy of the Greek mathematician Diophantos' *Arithmetic*, now, as in Fermat's day, recognized as an early work in Number Theory. Problem 8 from Book II of that work shows how to solve the equation $x^2 + y^2 = z^2$ for rational $x,\ y,\ z$. (This is in fact a simple generalization of the case in which all of $x,\ y,\ z$ are positive integers.) In the margin of his copy of this work, Fermat wrote: "But it is not possible to split a cube into two cubes or a fourth power into two fourth powers or in general any of the infinitely many powers larger than two into two like terms. I have discovered a truly wonderful proof of this fact, but the margin is not large enough to include it."

When Fermat died (1661), this quotation came to light, and led to centuries of effort in order to produce a proof. Eventually one was produced in 1995 by (now Sir) Andrew Wiles. That proof itself is over 100 pages long and took Wiles seven years to produce. It uses concepts that were certainly not available to Fermat. The consensus is that Fermat's "proof" must have been defective. However, Hadamard is less concerned over this point than with the intuition that led Fermat to state the proposition in the first place. It really is a truly remarkable statement. Among other things, it is an early example of a statement of the non-existence of an entire class of solutions. There are very few such results in Diophantos' surviving text.

Hadamard's second example concerns the mathematician Bernhard Riemann (1826–1866). On Riemann's death there was found among his papers a note listing a number of properties of a function that he called $\varsigma(s)$, now known as the Riemann zeta-function in his honor. One of these properties, to do with the positions of the function's zeros, holds particular interest. This is known as the Riemann Hypothesis. I gave a brief account of it in my *Parabola Incorporating Function* column in October 2004. Now, almost a century and a half later, it is still unproved, although it is almost universally believed to be true. It is the major unproved conjecture of today's Mathematics and is central to much of modern Number Theory. Riemann had written: "These properties of $\varsigma(s)$ are deduced from an expression for it which, however, I did not succeed in simplifying enough to publish."

Hadamard comments: "We still have not the slightest idea of what the expression could be. As to the properties he simply enunciated, some thirty years elapsed before I was able to prove all of them but one."

That one was, of course the Riemann Hypothesis itself.

The third of Hadamard's examples concerns the mathematician
Évariste Galois (1811 − 1831). Yes! Those dates are correct. Galois, one of the greatest mathematicians of all time, died at the age of twenty, killed in a duel. The night before he died, and evidently knowing that that he was facing almost certain death,

he wrote to a friend outlining some results he claimed, asking that they be brought to the attention of prominent mathematicians of the day. Galois is remembered today as the instigator of a branch of Algebra (now known as Galois Theory), but this is not the thing that primarily interests Hadamard. Rather he draws attention to other results, to do with the properties of certain integrals. It was to be a quarter of a century before these results were proved, and that by techniques Galois would very probably not have known.

The final entry on Hadamard's list is an intuition by Poincaré. In one of his books, he uses a result, as if it were well known, that actually did not have this status. It had in fact just been derived, but in a paper Poincaré could not possibly have seen.

All four of these stories have in common the feature that a result is produced without proof, and (at least in the first three) when the techniques by which they were ultimately derived had not yet been developed. The fourth story is perhaps not so surprising. The result that Poincaré used is the sort of thing he could well have derived himself by the means at his disposal; he could easily have believed that others had got there before him, and so have believed it "well-known". There is, of course, one other exception to the general description I have just given: the Riemann Hypothesis, which still eludes us.

But the first three (at least) of these stories illustrate very strikingly the result of what I am calling a "subliminal deduction". Because no proofs were given by Fermat, Riemann or Galois, we have no way of knowing if their chains of subliminal reasoning were valid or not, but the remarkable fact is that the results were (with the just possible exception of the Riemann Hypothesis) all true!

The fourth story, as I have indicated, stands somewhat apart. However, for this very reason it perhaps provides a purer example of subliminal deduction than do the others. Because it was the sort of thing he could have done with the techniques at his disposal, it seems very likely that Poincaré did deduce the result in question, but never brought to consciousness the route he had taken.

Although it lies outside Hadamard's main concerns, I will repeat also another tale he tells. He learned it from the American number theorist L. E. Dickson ($1874 - 1954$), and concerns Dickson's mother. Here as Hadamard tells it, is the story.

> [She] and her sister, who, at school, were rivals in geometry, had spent a long and futile evening over a certain problem. During the night, his mother dreamed of it and began developing the solution in a loud and clear voice; her sister, hearing that, arose and took notes. On the following morning in class, she happened to have the right solution which Dickson's mother failed to know.

This story perhaps takes me a little away from my main purpose in this article, but I find it interesting (as did Hadamard), so interesting that I cannot resist repeating it. In terms of Hadamard's "stages", it follows the main outline until stage 3, the "illumination", which remains in the subconscious mind, as indeed does stage 4, the "verification".

I will close with a speculation, and perhaps risk the ire of some sections of the feminist movement. It is now well established that male and female brains are wired differently, although it is quite unknown if this affects the way in which males and females do Mathematics, if in fact there is *any difference at all* in this regard.[3]

However, outside Mathematics, when it comes to subliminal deduction, may we not have here the key to what is widely accepted as "feminine intuition"? While in Mathematics, the route to a discovery is important and must be given, in other areas of human endeavour this is not so; what matters is the result itself. Go back to my first story, the one about the boy who fell down stairs. Here we have no need to know *how* the wife deduced the child's misfortune; what mattered was that she *did*.

Finally, carry out a thought experiment and imagine the father and the mother swapping places, so that it was the father who went to the dentist, and the mother who stayed home. Do you think the father would have rushed home from the dental chair?

[3]Contemporary research supports the conclusion that boys and girls bring to Mathematics, the same approaches and the same levels of ability. It was once common to believe otherwise. For example the mathematician Mary Somerville (1780-1872) was discouraged in her mathematical interests by her father, a naval officer, because he feared that "the strain of abstract thought would injure the tender female frame". However, quite clearly it did her no harm at all! She remained mentally alert almost to the day of her death, a little short of her 92nd birthday.