Parabola Volume 47, Issue 3 (2011)

Solutions 1361–1370

Q1361 Find a six-digit number which can be split into three two-digit squares and also into two three-digit squares. (The first digit of a number cannot be zero.)

SOLUTION The number must begin with a three–digit square whose first two digits also form a square. So we seek a three–digit square of the form

$$16\text{X} \quad \text{or} \quad 25\text{X} \quad \text{or} \quad 36\text{X} \quad \text{or} \quad 49\text{X} \quad \text{or} \quad 64\text{X} \quad \text{or} \quad 81\text{X} \quad ;$$

the possibilities are 169, 256 and 361. The last of these digits must begin a two–digit square, which rules out 169. The remaining options for our six–digit number are

$$2564\text{XY} \quad \text{and} \quad 3616\text{XY} .$$

Now 4XY is a three–digit square beginning with 4, and so we have $\text{XY} = 00, 41$ or 84; the first is ruled out by the conditions of the problem and the others are not squares. The only answer to the problem is 361625.

Q1362 Sandy leans a ladder against a wall in order to clean the gutter running along the top of the wall. Sandy is worried that the foot of the ladder is going to slip away from the wall and therefore ties a tightly stretched string between the middle of the ladder and a nail which is located directly below the top of the ladder, at the point where the floor meets the wall. Assuming that the floor is perfectly horizontal and the wall is perfectly vertical, how much is this going to help?

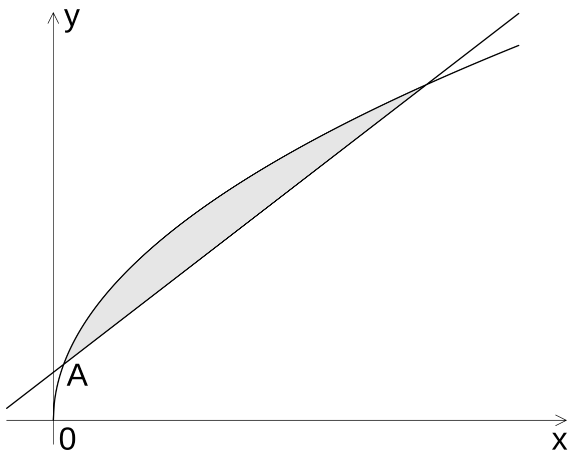
SOLUTION If the foot of the ladder slips away from the wall then the middle of the ladder is always the same distance from the nail. (Why? Because the angle between the wall and the floor is a right angle, so the line from the nail to the middle is always a radius of the circle having the ladder as diameter.) So connecting these two points by a string is not going to help at all!!

Q1363 Find the smallest possible value of $x^2 + y^2$, if x and y are real numbers for which $y \geq 2 + 3x$ and $y \leq 7\sqrt{x}$.

SOLUTION The graphs of $y = 2 + 3x$ and $y = 7\sqrt{x}$ are shown in the diagram, and the allowable values of the pair (x, y) lie in the shaded region. As $x^2 + y^2$ is the square of the distance from the origin to (x, y), it is clear that its smallest value occurs at the point marked A. Solving the equations simultaneously, we have

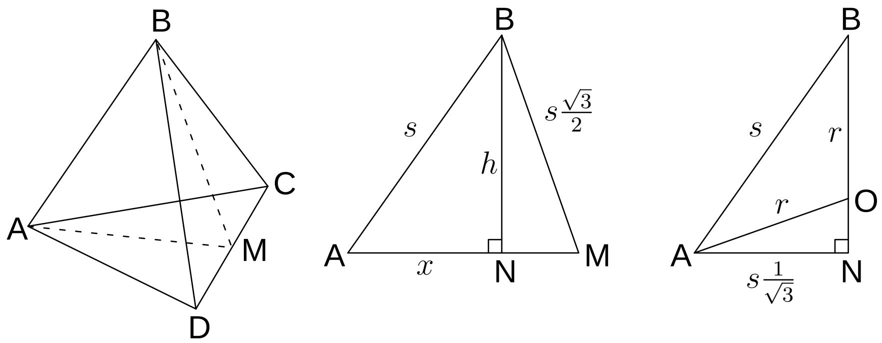
$$\begin{aligned} 2 + 3x = 7\sqrt{x} \quad &\Rightarrow \quad (2 + 3x)^2 = 49x \\ &\Rightarrow \quad 9x^2 - 37x + 4 = 0 \\ &\Rightarrow \quad (9x - 1)(x - 4) = 0 \\ &\Rightarrow \quad x = \tfrac{1}{9} \text{ or } x = 4 \, . \end{aligned}$$

At A we have $x = \frac{1}{9}$; therefore $y = 2 + 3x = \frac{7}{3}$ and the minimum value of $x^2 + y^2$ is $\frac{442}{81}$.



Q1364 Find the total surface area, and the volume, of a regular tetrahedron inscribed in a sphere of radius r.

SOLUTION It is easiest first to find the surface area and volume in terms of the side length of the tetrahedron, and then relate this to the radius. Let the side length be s, and consider the following diagrams.



In the first diagram M is the midpoint of CD. Therefore AM and BM are altitudes of equilateral triangles and they have length $s\sqrt{3}/2$. Applying Pythagoras' Theorem to the two right–angled triangles in the second diagram gives

$$x^2 + h^2 = s^2 \quad \text{and} \quad \left(s\,\frac{\sqrt{3}}{2} - x\right)^2 + h^2 = \left(s\,\frac{\sqrt{3}}{2}\right)^2 .$$

Now expanding the second equation yields

$$\frac{3}{4}\,s^2 - sx\sqrt{3} + x^2 + h^2 = \frac{3}{4}\,s^2 .$$

Simplifying and using the first equation above, we have

$$-sx\sqrt{3} + s^2 = 0$$

and hence

$$x = \frac{s}{\sqrt{3}}\ , \qquad h = \sqrt{s^2 - \Bigl(\frac{s}{\sqrt{3}}\Bigr)^2} = s\sqrt{\frac{2}{3}}\ .$$

Next consider the third diagram, which is just the left part of the second diagram, relabelled. The centre of the sphere will be the point O on BN such that OA is equal to OB (and is the radius of the sphere). Pythagoras' Theorem in $\triangle AON$ gives

$$\begin{aligned} r^2 &= \Bigl(\frac{s}{\sqrt{3}}\Bigr)^2 + (h-r)^2 \\ &= \frac{s^2}{3} + h^2 - 2rh + r^2 \\ &= s^2 - 2rs\sqrt{\frac{2}{3}} + r^2 \\ s &= 2r\sqrt{\frac{2}{3}} \end{aligned}$$

Finally, each face of the tetrahedron is an equilateral triangle and has area

$$A = \frac{1}{2}\,s\Bigl(s\frac{\sqrt{3}}{2}\Bigr) = \frac{\sqrt{3}}{4}\,s^2 = \frac{2}{\sqrt{3}}r^2\ .$$

So the total surface area and volume are given by

$$S = 4A = \frac{8}{\sqrt{3}}r^2 \quad \text{and} \quad V = \frac{1}{3}Ah = \frac{2}{3\sqrt{3}}r^2\Bigl(2r\sqrt{\frac{2}{3}}\Bigr)\sqrt{\frac{2}{3}} = \frac{8}{9\sqrt{3}}r^3\ .$$

Q1365 Show that $9999999999999999999999999999999991$ is not prime.

SOLUTION We have

$$\begin{aligned} &9999999999999999999999999999999991 \\ &\qquad = 10^{34} - 9 \\ &\qquad = (10^{17})^2 - 3^2 \\ &\qquad = 99999999999999997 \times 100000000000000003 \end{aligned}$$

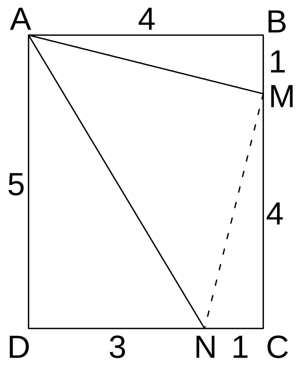
which is composite.

Comment. As it turns out, both of the factors on the right-hand side are prime – this is not a simple calculation, but you don't need to do it in order to answer the question.

Q1366 Draw a rectangle $ABCD$ with side lengths $AB = CD = 4$ and $AD = BC = 5$. Let M be a point on BC with $BM = 1$, and N a point on CD with $CN = 1$. Use this diagram to prove that

$$\frac{\pi}{4} = \arctan\frac{1}{4} + \arctan\frac{3}{5}\ .$$

SOLUTION



From the diagram we have

$$\angle BAM = \tan^{-1}\frac{1}{4} \quad \text{and} \quad \angle DAN = \tan^{-1}\frac{3}{5}\,.$$

Also, $\angle AMN$ is a right angle and $AM = MN$; so $\triangle AMN$ is isosceles and $\angle MAN = \pi/4$. Thus

$$\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{3}{5} + \frac{\pi}{4} = \frac{\pi}{2}\,,$$

and the result follows.

(Problem suggested and solved by Edward Lee, a first year student at UNSW.)

Q1367 If n is a positive integer and p is a prime, we write $\nu(p, n!)$ for the exact power of p which is a factor of $n!$: that is, p^ν is a factor of $n!$ but $p^{\nu+1}$ is not. For example, $\nu(3, 10!) = 4$ because 3^4 divides $10!$ but 3^5 does not. Prove that

$$\nu(p, n!) = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots\,.$$

Here the brackets $\lfloor\ \rfloor$ indicate that the number inside is to be rounded down, for example, $\lfloor \pi \rfloor = 3$.

SOLUTION We have to count the total number of factors of p in $n!$. Every pth number is a multiple of p, and so we count $\lfloor n/p \rfloor$ factors. However every p^2th number is a multiple of p^2 and contains two factors of p; we have already counted one of them and so we have to count another one in each case, an additional $\lfloor n/p^2 \rfloor$ factors. Proceeding in the same way for multiples of p^3 and so on completes the proof.

Q1368 If $N = 2011!$, how many hundredth powers are factors of N?

SOLUTION We begin by using the result of the previous question with $n = 2011$. Calculation gives

$$\begin{aligned}
\nu(2, 2011!) &= 1005 + 502 + 251 + 125 + 62 \\
&\qquad + 31 + 15 + 7 + 3 + 1 = 2002 \\
\nu(3, 2011!) &= 670 + 223 + 74 + 24 + 8 + 2 = 1001
\end{aligned}$$

$$\begin{aligned}\nu(5,2011!) &= 402+80+16+3=501\\ \nu(7,2011!) &= 287+41+5=333\\ \nu(11,2011!) &= 182+16+1=199\ .\end{aligned}$$

To save a bit of work we note that $\nu(p,n!)$ always decreases as p increases; so for $p = 13, 17, 19$ we have

$$100 \le \left\lfloor \frac{2011}{p} \right\rfloor \le \nu(p,2011!) \le 199\ .$$

Finally,

$$\nu(23,2011!) = 87+3 = 90$$

and so $\nu(p,2011!) < 100$ if $p \ge 23$. Now as in the solution to problem 1356 (see the previous issue of *Parabola*), the largest hundredth power which is a factor of $2011!$ is

$$\left(2^{20}\times 3^{10}\times 5^5\times 7^3\times 11\times 13\times 17\times 19\right)^{100}$$

and the number of hundredth powers which are factors of $2011!$ is the number of factors of the expression in brackets. A factor of this number is

$$2^a\,3^b\,5^c\,7^d\,11^e\,13^f\,17^g\,19^h$$

where a is $0,1,2,\ldots$ or 20 and b is $0,1,2,\ldots$ or 10 and $\ldots$ and h is 0 or 1. So our final answer is

$$21\times 11\times 6\times 4\times 2\times 2\times 2\times 2 = 88704\ .$$

Q1369 Prove that if

$$a+\sqrt{b} = c+\sqrt{d}\ ,$$

where a, b, c and d are rational numbers and $\sqrt{b}$ and $\sqrt{d}$ are irrational, then $a = c$ and $b = d$.

SOLUTION Squaring both sides, we have

$$a^2+b+2a\sqrt{b} = c^2+d+2c\sqrt{d}\ .$$

Subtracting this equation from $2c$ times the given equation,

$$2ac-a^2-b-2(a-c)\sqrt{b} = c^2-d\ .$$

It follows that $a-c=0$; for if not then we would have

$$\sqrt{b} = \frac{2ac-a^2-b-c^2+d}{2(a-c)}\ ,$$

contradicting the fact that $\sqrt{b}$ is irrational. So $a = c$, and then it is easy to see that $b = d$.

Correct solution received from Colin A. Wilson, Victoria.

Q1370 Find the smallest positive integer a for which the surd

$$\sqrt{a + 20\sqrt{11}}$$

can be simplified as $\sqrt{x} + \sqrt{y}$, where x and y are positive integers.

SOLUTION We need

$$a + 20\sqrt{11} = \left(\sqrt{x} + \sqrt{y}\right)^2 = x + y + 2\sqrt{xy}\ ;$$

since $\sqrt{11}$ is irrational so is $\sqrt{xy}$, and the previous question shows that

$$x + y = a \quad \text{and} \quad xy = 1100\ .$$

So we need to find the minimum value of $x + y$, where x and y are positive integers whose product is 1100. This is achieved by taking x and y as close together as possible, so $x = 25$ and $y = 44$ (or *vice versa*). Hence the smallest possible a is 69, and we have

$$\sqrt{69 + 20\sqrt{11}} = \sqrt{25} + \sqrt{44} = 5 + 2\sqrt{11}\ .$$

Correct solution received from Colin A. Wilson, Victoria, who also pointed out that there are only finitely many a such that the given expression can be simplified as desired.