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History of Mathematics: The Population Explosion

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The title of this column is a phrase that became a catchcry of the 1960s, and although the problem that it encapsulates is still very much with us (indeed even more acute than it was back then), it is less in vogue nowadays. The phrase refers to the unprecedented growth of the human population that is currently taking place. It is particularly apposite to look once again at the issue involved, because as I write (2011) the world population has just topped 7 billion (7000 million).

It will be convenient to begin our account of the underlying mathematics with the work of Thomas Malthus (1766-1834), who in 1798 produced a most influential book, *An Essay on the Principle of Population*. It ran in all to six editions in Malthus's lifetime and has been much reprinted since. Among other things it is seen as a pioneering contribution to economics.

The principal point that Malthus was concerned to make was that "Population, when unchecked, increases in a geometrical ratio". This was announced on p. 14 of his first edition. Later on in that same edition (p. 21) he was more explicit: "Population, when unchecked, goes on doubling itself every twenty-five years".

We can analyze this situation by using a little mathematical notation. If we start with an initial population of N_0 and suppose that it grows at $R\%$ per year, then after n years, it will be[2] $N_0\,(1+R/100)^n$. So, in mathematical language, Malthus said that $N_0\,(1+R/100)^{25} = 2N_0$. We may solve this equation for R, and the easiest way to do this is to take logarithms.[3]

$$\begin{gathered}
\log N_0 + 25\log(1+R/100) = \log 2 + \log N_0 \\
25\log(1+R/100) = \log 2 \\
\log(1+R/100) = \tfrac{\log 2}{25} \approx 0.012 \\
1 + \tfrac{R}{100} \approx 10^{0.012} \approx 1.028
\end{gathered}$$

Thus Malthus was saying that the population grows, if unchecked, at a rate of 2.8% per annum.[4] This rate may well have been the rate prevailing in the Britain of his day. At that time, Britain was experiencing the effects of the industrial revolution, but these

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[2]Readers may recall that this is the same formula that is used to calculate compound interest.

[3]Here I use common (base 10) logarithms. It is a straightforward matter to perform the same calculation (and, of course, reach the same answer) using natural logarithms. This I leave to the reader.

[4]For relatively low interest rates, there is an approximate formula for R in this equation: $R \approx 72/n$, where n is the number of years required for doubling the initial value. This approximation was the subject of two articles in *Function* (October 1988 and June 1997).

had not then spread to all other parts of the globe. There have been several attempts to estimate the world population in bygone days. They vary considerably and most have been posted on the web. I will use the one at[5]

http://worldhistorysite.com/population.html

This gives an estimated population of the world as being 900 million in 1800 and 1200 million in 1850, corresponding to an annual growth-rate of 0.57%. So a figure somewhat like this may be somewhat more realistic than Malthus's one as being applicable to the world as a whole.

Even so, the geometric increase, as a description of what has actually happened in history, cannot be accurate. There are several somewhat spectacular illustrations of this. If we begin with a single couple in the year 4000 BCE[6] and suppose an annual increase of only 0.5%, this gives a total population of about 2×10^{13} individuals that would be alive today. The entire surface of the earth (land and water!) is about 5×10^{8} square kilometres, which means that we would now be packed in at a density greater than 40,000 people per square kilometre. Luckily we have not yet reached this level of overcrowding. Even the city of Mumbai, one of the most densely populated areas in the world, has only 34,000 people per square kilometer.

The calculation just performed has been given in many forms. The result depends critically on the rate of increase assumed and also on the time over which the population increase is supposed to have occurred. In more bizarre scenarios, the earth becomes a solid sphere of human flesh extending beyond the orbit of Mars!

Another way of looking at things produces yet another paradox. If we take the present (2011) figure for world population as 7 billion, and compare it with the figure, 6 billion, given at the website detailed above for 1999, we find an annual rate of increase of 1.29%. Now proceed backwards and estimate when, on this figure, the human race began. We have $N_0 = 2,\ R = 1.29$, and we use our formula to calculate n. That is we solve for n the equation

$$2 \times 1.0129^{n} = 7,000,000,000.$$

I leave readers to provide the details, but the solution to this equation is $n \approx 1715$, that is to say, humans would have been around for somewhat less than two millennia! This too is unrealistic in the extreme.

These two paradoxes show that we need to pay more attention to what Malthus actually said. The key phrase is "if unchecked".

Let us think in a little more detail about what is involved. Suppose that, at some time t, the population is N, and at some later time $t+\Delta t$ shortly thereafter it is $N+\Delta N$. (The symbol Δ may usefully be thought of as standing for "the change in".) The

[5] Accessed 8/11/2011.

[6] It was once popular to assume this. The assumption was based on a literal reading of the Book of Genesis, which was taken to be true in every particular detail. Sadly, but perhaps not altogether surprisingly, there are websites that still cling to this methodology, at a time when all reputable theologians have discarded it.

change in N is brought about by two processes: birth and death. Let the birthrate be B and the deathrate be D. Then in the time Δt, $BN\Delta t$ individuals will be born and $DN\Delta t$ will die. The upshot is that $\Delta N = (B - D)N\Delta t = rN\Delta t$, say. (The quantity r is now known as the *Malthusian parameter* in honor of Malthus.) Now suppose that the increment in time Δt is very short. In that case, the governing equation just derived can be approximated by one involving a derivative:

$$\frac{dN}{dt} = rN.$$

In the case where r is constant, this equation has the solution

$$N = N_0 e^{rt}.$$

This is the equation of exponential increase, which is the same thing as Malthus's "geometrical ratio".

Now Malthus was acutely aware that such an exponential increase could not be sustained. He stated that the production of food ("sustenance") could only (at best) increase linearly, so that the exponential increase in the population would come to outpace the production of food.[7] The result would be widespread famine and human misery. (This prediction led to economics being labeled "the dismal science".) Thus Malthus advocated a reduction in the birthrate in order to prevent this eventuality.

The Malthusian parameter r is the difference between the birthrate B and the deathrate D. Exponential growth necessarily occurs if $r > 0$, i.e. if $B > D$. For most of human history, birthrates and deathrates were both high, with perhaps the birthrate being very slightly the greater. If we take estimates of the population as given by the website quoted earlier, we find that the human population is supposed to have risen from 4 million to 5 million in a space of 5000 years. This is a growth rate of about 0.01%, which we can regard (given all the uncertainties in the data) as essentially zero. The origins of the human race predate 10,000 BCE. The first recognizable modern humans (*Homo sapiens*) probably date from about 100,000 BCE, so that there were 90,000 years over which the numbers rose to perhaps 4,000,000, and thus also at a rate somewhere near 0.01% This too is a very slow rate of increase. We should compare these figures with those prevailing today. According to the website noted above, the human population in 1700 was about 610 million. Now (2011) it is 7000 million. This is an average growth rate for this period of just under 0.08%. Clearly, this is a major change from the earlier 0.01%. Indeed, the Malthusian parameter has been increasing (overall) between those very early times and our modern era.

With the rise of modern civilization, deathrates fell, but with birthrates remaining high, the Malthusian scenario became a real concern.

Figure 1 illustrates this especially vividly. The steep rise at the right-hand edge shows how apt the term "explosion" really is. Whereas for almost 12 millennia the population remained relatively small (so small that it hardly registers on the graph), the right-hand edge of the graph shows a spectacularly dramatic upswing.

[7] An even more obvious argument would have been to note that the available land area is a fixed constant.

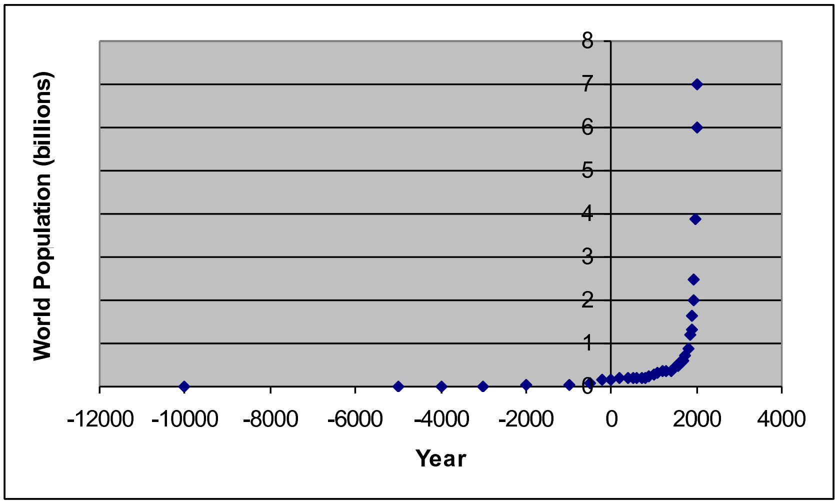

Figure 1: World Population from 12,000 BCE till present day

The calculations given above show that such increase cannot possibly be sustained, so the question (ultimately a *moral* question) is: "How is the stabilization of human numbers to be brought about?" What are the Malthusian "checks" that can be supplied? We have only two options available: decrease the birthrate or increase the deathrate. The latter option is precisely the one Malthus feared, with its obvious consequences of human suffering and misery. It follows that the only humane option is a decrease in the birthrate. This then is what Malthus advocated, although he believed that, with the methods available back in 1800, this would be difficult to achieve. Modern technology has put into our hands improved methods of birth control, and so there may be hope that the population may be restricted by their use, although there are still powerful voices opposing such means. The next few decades will surely show which scenario prevails.

Now that we have seen that population does not actually increase according to Malthus's law (except for very short periods of time), we may wonder if the law is actually much use at all! However, if we look at it as not so much a description of historical fact as an identification of an *underlying tendency*, then we get a better appreciation of its purpose. The two founders of modern evolutionary theory, Charles Darwin and Alfred Wallace, were both much influenced by Malthus's book. It led them to realize that the individual members of a (non-human) population must compete for places in the ecosystem they inhabit and it is those that are best adapted to their environment who will survive.

When it comes to the question of producing a formula that seeks to capture the actual course of the historical events, there have been several attempts. I will briefly describe three. The first was the work of the Belgian scientist Pierre François Verhulst (1804-1849), but it became better known as a result of later and independent study by the US biologist Raymond Pearl (1879-1940). Both these researchers replaced the simple equation of exponential growth, $\frac{dN}{dt} = rN$, with a modification

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right).$$

In this version, there is a second constant, K, as well as the Malthusian parameter r. K is termed the "carrying capacity", because it is the maximum value of N that the environment can support. Readers can very readily check this statement straight from the equation: when $N < K$, $\frac{dN}{dt} > 0$, and so the population will increase, but when $N > K$, $\frac{dN}{dt} < 0$, and so the population declines. If we begin with a value N_0 of N lying in the region $0 < N_0 < K$, then the value of N increases toward an asymptote at $N = K$. The equation just given is sometimes referred to by the names of either one or the other or both of its discoverers, but the more common term is "logistic". The curve depicting the values of N is known as the "logistic curve" and has the formula

$$N = \frac{KN_0}{N_0 + (K - N_0)\, e^{-rt}}$$

(The curve is often described as "sigmoid" or "S-shaped", and if we think of a very flattened, stretched out S, then so be it!)

For both Pearl and Verhulst, the initial interest focused on the theoretical reasons for preferring the somewhat more complicated differential equation, but Pearl in particular tried to match it to actual data. Although it displayed some initial success when he applied it to the US population for the years 1709-1910, it came badly unstuck later on. In particular, Pearl had $K = 200,000,000$; the current US population is in excess of $300,000,000$. Attempts were also made to replicate the model under laboratory conditions. The results were not (to my mind, at least) particularly convincing.

Nevertheless, this work has been influential, especially in the field of theoretical evolutionary theory. Biologists now routinely speak of "r-selection" and "K-selection". The former characterizes the evolutionary strategy of those organisms that produce enormous numbers of offspring of which only a few actually survive. Sea-turtles and many insect species provide examples. The latter strategy characterizes those species that adapt to better their survival in their ecological niches. Most mammals adopt this strategy.

My second example is a strange one. It was first mooted in 1960 by the electrical engineer Heinz von Foerster (1911-2002) and two colleagues, Patricia Mora and Lawrence Amiot. The first author was, among other things, a pioneer (with Norbert Wiener) of the science of cybernetics. His reputation was considerable. It is interesting to speculate whether the equation he proposed, along with its absolutely startling consequences, would ever have been published at all had it been the work of some lesser light.

However these authors proposed a different amendment to the Malthusian equation $\frac{dN}{dt} = rN$. Their equation incorporated a further constant $\epsilon\,(> 0)$. It was

$$\frac{dN}{dt} = rN^{1+\epsilon}.$$

This equation has the solution

$$N = \left(N_0^{-\epsilon} - \epsilon rt\right)^{-1/\epsilon}.$$

The extremely controversial aspect of this solution is that at a time $T = \frac{N_0^{-\epsilon}}{\epsilon r}$, the world population is set to become infinite! The three researchers fitted historical data to their formula and concluded that this would happen on Friday 13 November 2026. This date they called *doomsday*. So bizarre was the prediction that their article was widely believed to be a hoax, but they denied this, although they did say that, of course, the world population would not become infinite, but other catastrophes would intervene. The debate continued for at least ten years and there were even claims that we are actually ahead of schedule for doomsday.

I thought to test how we are going now. Readers who have followed me so far will realize that the attempts to fit formulae to historical data depend critically on the dataset we choose to use. For my test, I used, not the website quoted above, nor the complicated mix of population estimates employed by von Foerster and his coauthors, but rather a short summary supplied by the Wikipedia article *World Population*. This covers the time from Malthus to the present day. Here it is:

Population	Year
1 billion	1804
2 billion	1927
3 billion	1960
4 billion	1974
5 billion	1987
6 billion	1999
7 billion	2011

I fitted this dataset by both a Malthusian equation $N = Ae^{rt}$, and by a "doomsday equation" $N = B\,(T-t)^{-1/\epsilon}$. (Notice a couple of minor changes in notation.) Both gave excellent, although not perfect, fits. For the Malthusian equation, I found $A = 0.96$ (billion), $r = 0.009$ (9%), for the other, $B = 53.2,\ T = 2032,\ \epsilon = 1.54$. Goodness of fit is measured by means of a statistic known as the correlation coefficient. Perfect fit corresponds to a correlation coefficient of 1. My curve-fits gave 0.964 for the exponential (Malthusian) fit and 0.979 for the doomsday (von Foerster) one. Although the latter is (marginally) better, this is hardly significant as there are three parameters available for adjustment here as opposed to only two for the former. As a general rule, an increase in the number of assignable parameters would be expected to improve the fit.

Readers will note that on my calculations, doomsday is delayed for six years. I am most unlikely to live to see it, but readers should be prepared!

The name "doomsday" has also been applied to another analysis. This one is a joint enterprise by Brandon Carter, an eminent physicist, and John A. Leslie, a philosopher. The work of these two has now totally eclipsed the earlier study by von Foerster and his co-workers, but it is much harder to characterize. Leslie in particular has written on it at great length, but hardly in simple terms. He employs an intricate statistical argument that is far from being widely accepted, and reaches a conclusion that is equally problematical. Rather than giving the details of an extremely intricate and highly dubious calculation, I will simply quote the summary provided by Wikipedia.

> "If Leslie's Figure ... is used, then 60 billion humans have been born so far, therefore there is a 95% chance that the total number of humans N will be less than 20×60 billion $= 1.2$ trillion. Assuming that the world population stabilizes at 10 billion and a life expectancy of 80 years, it can be estimated that the remaining 1140 billion humans will be born in the next 9120 years. Depending on the projection of world population in the forthcoming centuries, estimates may vary, but the main point of the argument is that it is unlikely that more than 1.2 trillion humans will ever live."

The part of this quotation that I have left out is a reference to what should be the source of the figure of 60 billion. In fact it is nothing of the sort. (It is an article co-authored by a Monash colleague of mine, which does indeed cite the 60 billion figure, but actually expresses a mild skepticism about it!) I have searched a lot of Leslie's writing (although by no means all; there is a great deal of it), but without any success in discovering where his 60 billion estimate comes from.

However, I did find another website.[8] This is an estimate by Carl Haub, an American demographer, which yields an even higher figure than Leslie's, over 100 billion. I tried to calculate this number myself some years ago, and came up with a much lower figure, about 10 billion as of April 1990. (See *Function* for this date.) My figure was so much lower for three reasons: first, it was calculated over 20 years ago and a lot more people have been born since; secondly, I took a much lower estimate of prehistoric numbers (1 million as against several millions); but thirdly because of a fundamental oversight. My calculation completely overlooked infant and child mortality. My way of counting people ignored those (many, in fact) who did not live long enough to reproduce. So I will conclude this article by recalculating what I think is a reasonable "guesstimate" of the total number of humans ever.

Suppose that at time t, there are $N(t)$ people on earth and that the birthrate is $B(t)$. Then the total number of people ever is exactly equal to the total number of people ever born, i.e., the total number of births. This is $\int_a^T B(t)N(t)dt$, where *a* is the date when the first humans appeared (which I take to be about $100,000$ BCE) and T is the present date (2011, as I write). Haub calculates an approximation to this integral by breaking up the time into ten unequal intervals, and giving estimates of the mean values $\overline{N}$ of $N(t)$ and $\overline{B}$ of $B(t)$. Then for the interval $(\tau, \tau + \sigma)$, say, the number of births during that interval is estimated to be $\overline{B}\,\overline{N}\sigma$. The total number of births ever is then the total of the contributions from all ten intervals.

He starts his clock at 50,000 BCE, whereas I start at 100,000 BCE. I took my estimates of population from the website listed earlier. Generally, their estimates of $N(t)$ are lower than Haub's. But I assume a $B(t)$ value of 0.05 for most of human history. This is the value prevailing in the poorest countries today, and I doubt that Haub's figure, 0.08, for the earliest epochs is realistic. However, I kept the figure of 0.05 for one interval (1850-1900), whereas Haub has 0.04. It seemed to me that before 1900,

[8] http://www.prb.org/Articles/2002/HowManyPeopleHaveEverLivedonEarth.aspx Accessed 8/11/2011.

most of the world would have had no access to effective birth control, and that child mortality would also have likely been high in most places, giving a strong incentive to high birthrates.

The upshot was that my "guesstimate" of the total number of humans ever was about 60 billion, essentially the same figure as that given by Leslie. This implies that something like 12% of all the people who ever lived are alive at this moment. This figure is set to rise, indeed it is rising daily, but it cannot possibly reach 50%. It could only do so if the living population exceeded 30 billion, and this is seen as quite unrealistic.

I began my 1990 article by quoting a Tasmanian boy's answer to a religion teacher who asked him how he knew he would die: "Well, most people have so far". Back then, I cast some doubt on this. But it seems he was right after all!