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How to derive the quadratic formula

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Introduction

The solution formula to the quadratic equation

$$ax^2 + bx + c = 0 \tag{1}$$

is usually derived in textbooks by completing the square. This is done in the following way (see [1]):

"When you use the technique of completing the square to solve quadratic equations, begin by rewriting the equation in the form $x^2 + \frac{b}{a}x = -\frac{c}{a}$. Next, add $\left(\frac{1}{2} \cdot \frac{b}{a}\right)^2$ to both sides of the equation, and then express the variable side as a square. Take the square root of both sides, and solve the two resulting linear equations for x."

This is very unnatural and potentially confusing for students who see it for the first time. The following approach is more appropriate:

Multiply both sides by $4a$.	$ax^2 + bx + c = 0$
Add $b^2 - 4ac$ to both sides.	$4a^2x^2 + 4abx + 4ac = 0$
Factor the left side as a perfect square.	$(2a)^2x^2 + 2(2a)bx + b^2 = b^2 - 4ac$
	$(2ax + b)^2 = D$

where the expression $D = b^2 - 4ac$ is the discriminant of the quadratic equation (1). This gives the well-known solution formula to (1):

$$x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\,. \tag{2}$$

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More detailed explanations

The key to this technique is the identity

$$(Ax+B)^2 = A^2x^2 + 2ABx + B^2$$

We see that the middle term of the quadratic trinomial should contain the number 2.

$$(Ax+B)^2 = A^2x^2 + \overset{\Downarrow}{2}ABx + B^2$$

For this reason, we multiply the quadratic equation (1) by the number 2.

$$\text{Multiply both sides by } 2\,. \qquad \begin{aligned} ax^2 + bx + c &= 0 \\ 2ax^2 + 2bx + 2c &= 0 \end{aligned} \tag{3}$$

Next, we want the quadratic coefficient to be a square.

$$(Ax+B)^2 = \overset{\Downarrow}{A^2}x^2 + 2ABx + B^2$$

For this reason, we multiply the equation (3) by the expression $2a$.

$$\text{Multiply both sides by } 2a\,. \qquad \begin{aligned} 2ax^2 + 2bx + 2c &= 0 \\ (2a)^2x^2 + 2(2a)bx + 4ac &= 0 \end{aligned} \tag{4}$$

From there, we see that it will be appropriate to choose $A = 2a$ and $B = b$. Finally, we need to have b^2 ($= B^2$) instead of $4ac$ in the equation (4).

$$\begin{aligned} &\text{Add } b^2 \text{ to both sides.} & (2a)^2x^2 + 2(2a)bx + 4ac &= 0 \\ &\text{Subtract } 4ac \text{ from both sides.} & (2a)^2x^2 + 2(2a)bx + b^2 + 4ac &= b^2 \\ &\text{Factor the left side as a perfect square.} & (2a)^2x^2 + 2(2a)bx + b^2 &= b^2 - 4ac \\ && (2ax+b)^2 &= b^2 - 4ac \end{aligned}$$

Final notes

This method (in the form as stated in the introduction) can be found in [2], or online

https://www.quora.com/What-is-the-best-way-to-explain-the-quadratic-formula

but, unfortunately, not in textbooks. You can even buy a T-shirt with it:

https://www.redbubble.com/i/t-shirt/Quadratic-Formula-Derivation-by-MathShirts/40392693.NL9AC

Finally, let us note that the following identity

$$(2ax+b)^2 - 4a(ax^2+bx+c) = b^2 - 4ac$$

from [3] can be derived in a similar way:

Multiply both sides by $4a$.	$ax^2+bx+c =: p(x)$
Add b^2-4ac to both sides.	$4a^2x^2+4abx+4ac = 4ap(x)$
Factor the left side.	$(2a)^2x^2+2(2a)bx+b^2 = 4ap(x)+b^2-4ac$
Subtract $4ap(x)$ from both sides.	$(2ax+b)^2 = 4ap(x)+b^2-4ac$
	$(2ax+b)^2-4ap(x) = b^2-4ac$

All you now have to do is substitute for $p(x)$.

References

[1] S.L. McCune, *Algebra I. Review and Workbook*, McGraw-Hill Education, 2019.

[2] B. Pieronkiewicz and J. Tanton, Different ways of solving quadratic equations, *Annales Universitatis Paedagogicae Cracoviensis, Studia ad Didacticam Mathematicae Pertinentia* **11** (2019), 103–125.

[3] J. Zimba, A short derivation of the quadratic formula, *Parabola* **59(2)** (2023), 6 pages.