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Games of Nim with restriction

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1 Introduction and games of Nim

In this study, we investigate Nim games, which are combinatorial games. A combinatorial game is one of the best themes for high school mathematics research, because there are still many unstudied themes.

Research on combinatorial games began approximately 120 years ago with the work of Prof. Charles L. Bouton of Harvard University [2], but research on combinatorial games was popularized when personal computers became widely available approximately 40 years ago.

Compared to research on other branches of mathematics, combinatorial games comprise a very fresh field of study. Dr. Miyadera, one of the authors of the present article, has studied combinatorial games with his high school students, and they have published 15 articles on the topic, such as [7], [10], [3] and [11].

This article aims to introduce high school and undergraduate students to research on combinatorial games, enabling them to undertake research without consulting any books. Upon selecting topics similar to the problems in this article, they can replicate the methods of proofs as well. Some proofs are slightly complicated; however, the difficulties lie in the symbols and formulas, whereas the actual methods themselves are not difficult. We use some illustrations to help readers of this article understand the proof.

In Section 2, we discuss a well-known game of two-pile Nim and the corresponding winning strategies of players. Moreover, we introduce the basic theory of winning the game.

In Section 3, we discuss restricted Nim. Subsection 3.1 describes the restricted Nim of one pile, which is a well-known game. We present the proof in an easily comprehensible manner using graphs. Subsection 3.2 discusses the restricted Nim of one pile with two types of stones proposed by a high school student, Shoei Takahashi. The proofs in this subsection are slightly difficult for high school students, and we use graphs to explain the methods of proof. Subsection 3.3 discusses a new restricted Nim of one pile, which was proposed by a high school student, Keita Mizugaki.

The Nim game is played with stone piles. A player can remove any number of stones from any one pile during their turn; the player who takes the last stone is considered the winner.

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Definition 1. For any position p of this game, a set of positions can be reached by precisely one move, which we denote as *move*(p).

Combinatorial games discussed in the present article are impartial games without draws; there will be only two kinds of positions.

Definition 2.

(a) A position is a *$\mathcal{P}$-position* if it is a winning position for the previous player (the player who just moved), as long as he/she plays correctly at every stage.

(b) A position is an *$\mathcal{N}$-position* if it is a winning position for the next player, as long as he/she plays correctly at every stage.

Theorem 3. *Any position on combinatorial games in the present article is either $\mathcal{P}$-position or $\mathcal{N}$-position, and not both.*

To understand the remainder of this article, it is adequate to accept Theorem 3 without proof. However, for a proof, see the work of Albert, Nowakowski and Wolfe [1]. Let $\mathbb{Z}_{\geq 0}$ and $\mathbb{N}$ denote the sets of non-negative integers and natural numbers, respectively.

2 Two-Pile Nim

We start with a two-pile Nim, which is a well-known game.

Definition 4. There are two piles of stones, and two players take turns removing stones from one of the piles. The player who takes the last stone or stones is considered the winner. We denote the position of the game by (x, y), where x and y are the numbers of stones in the first and second piles, respectively.

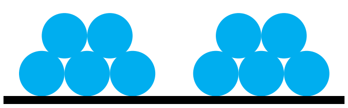

Figure 1: $(5, 5)$

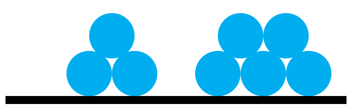

Figure 2: $(3, 5)$

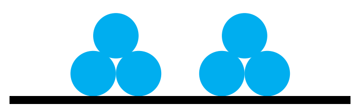

Figure 3: $(3, 3)$

Definition 5. Let

$$\mathcal{P} = \{(x, y) : x, y \in \mathbb{Z}_{\geq 0} \text{ and } x = y\}$$
$$\text{and} \quad \mathcal{N} = \{(x, y) : x, y \in \mathbb{Z}_{\geq 0} \text{ and } x \neq y\}.$$

Example 6. *The positions in Figures 1 and 3 belong to the set $\mathcal{P}$ in Definition 5, and the position in Figure 2 belongs to the set $\mathcal{N}$ in Definition 5.*

Lemma 7.

(i) Starting with $(x, y) \in \mathcal{P}$, one always moves to a position $(u, v) \in \mathcal{N}$.

(ii) Starting with $(x, y) \in \mathcal{N}$, one can always move to a position $(u, v) \in \mathcal{P}$.

Proof. (i) Assuming starting with $(x, y) \in \mathcal{P}$. By Definition 5, $x = y$, so from this position, any move leads to a position (u, v) such that $u \neq v$. Then, $(u, v) \in \mathcal{N}$.
(ii) Assuming starting with $(x, y) \in \mathcal{N}$. Then, by Definition 5, $x \neq y$. From this position, one can select a move that leads to a position (u, v) such that $u = v$. Then, $(u, v) \in \mathcal{P}$. □

Theorem 8. *In the game of Definition 4, the sets $\mathcal{P}$ and $\mathcal{N}$ of Definition 5 are the set of $\mathcal{P}$-positions and the set of $\mathcal{N}$-positions, respectively.*

Proof. Starting the game with a position in $\mathcal{P}$, by (i) of Lemma 7, any option leads to a position $(u, v) \in \mathcal{N}$. From this position, by (ii) of Lemma 7, our opponent can select an appropriate option that leads to a position $(u, v) \in \mathcal{P}$. Note that any option reduces some of the numbers in the coordinates. In this manner, the opponent can always reach a position $(u, v) \in \mathcal{P}$, and finally the opponent wins by reaching $(0, 0) \in \mathcal{P}$. Therefore, $\mathcal{P}$ is the set of $\mathcal{P}$-positions.

Starting the game with a position $(x, y) \in \mathcal{N}$, by (ii) of Lemma 7, one can select an appropriate option that leads to a position (p, q) in $\mathcal{P}$. From (p, q), any option selected by the opponent leads to a position in $\mathcal{N}$. In this manner, the game is won by reaching $(0, 0)$. Therefore, $\mathcal{N}$ is the set of $\mathcal{N}$-positions. □

3 Restricted Nim

There are several variants of the classical Nim game. Next, we investigate the Nim game with one pile. We must restrict the number of stones that players can remove in each turn; otherwise, the first player will take all the stones to win the game.

In Maximum Nim which is one of restricted games of Nim, we place an upper bound $f(n)$ on the number of stones that players can remove in terms of the number n of stones in the pile. Levine [5] provides an example of this game. Maximum Nim with three piles has also been studied by Miyadera and Manabe [8].

In the following, we investigate three restricted Nim games.

In the first game, we consider Maximum Nim, where the total number of stones to be removed is less than or equal to half of the number of stones in the pile. Miyadera et al. [6] provides a generalisation of this game.

In the second game, there are stones with a weight of 1 and stones with a weight of 2. Players can remove stones whose total weight is equal to or less than the ceiling of the half of the total weight in the pile. This game was proposed by Shoei Takahashi, who is one of the authors of the present article. The authors present formulas for the previous player's winning positions in this game. If we use "floor" instead of "ceiling", then the game becomes very difficult, and the proof of formulas for the previous winning position becomes complicated. Therefore, we use "ceiling" in this article.

In the third game, players can remove $f(k)$ stones at most in the k-th turn of the game, where f is a function whose values are natural numbers. This game was proposed by Keita Mizugaki, who is one of the authors of the present article. The authors present formulas for the previous player's winning positions in this game for $f(k) = mk$ with $m \in \mathbb{N}$.

The mathematical significance of this article is the introduction of two new games. Although the research results of these games are still in the early stages, the authors expect promising future research on these games.

From an educational perspective, this article provides possible avenues for mathematical research by high school students. By changing some parts of combinatorial games in this article and developing a computer program for the games, one can obtain data on the winning positions. Then, if a formula can be derived to describe the data, then a new game is discovered.

3.1 Maximum Nim

Definition 9. Assuming there is a pile of stones and two players take turns removing stones from the pile, when the total number of stones is $m \in \mathbb{N}$, a player is allowed to remove stones that are less than or equal to $\frac{m}{2}$. We denote a position in this game by (x), where x is the number of stones in the pile.

We define a move for this game. This is a concrete example of Definition 1. When there are m stones, then less than or equal to $\frac{m}{2}$ can be removed. Since the number of stones is a natural number, we can remove less than or equal to $\lfloor \frac{x}{2} \rfloor$ stones, which is the largest integer less than or equal to $\frac{x}{2}$.

Example 10. *Examples of the use of the floor function* $\lfloor \ \rfloor$ *include*
$\lfloor \frac{3}{2} \rfloor = 1,\ \lfloor \frac{9}{2} \rfloor = 4,\ \lfloor \frac{4}{2} \rfloor = \lfloor 2 \rfloor = 2$ *and* $\lfloor \frac{1}{2} \rfloor = 0.$

Definition 11. In the game of Definition 9, *move*$(x) = \{(x - t) : t \leq \lfloor \frac{x}{2} \rfloor\}$.

Definition 12. Let $\mathcal{P} = \{(2^n - 1) : n \in \mathbb{Z}_{\geq 0}\}$ and $\mathcal{N} = \{(i) : (i) \notin \mathcal{P}\}$.

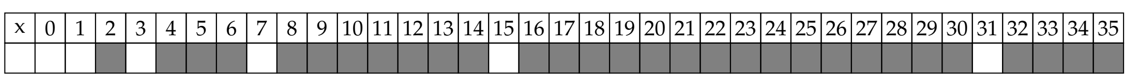

Figure 4: The positions in $\mathcal{P}$ are rectangles with white color

We will prove that $\mathcal{P}$ is the set of $\mathcal{P}$-positions. We have to prove that starting with a position in $\mathcal{P}$, one always moves to a position in $\mathcal{N}$, and starting with a position in $\mathcal{N}$, one can always move to a position in $\mathcal{P}$.

Since the positions in $\mathcal{P}$ are rectangles with white colour in Figure 4, we have to prove that we always move to a grey rectangle from a white rectangle, as shown in Figure 5, and we can always move to a white rectangle from a grey rectangle, as shown in Figure 6.

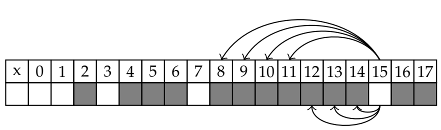

Figure 5: from $\mathcal{P}$ to $\mathcal{N}$

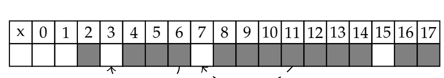

Figure 6: from $\mathcal{N}$ to $\mathcal{P}$

Example 13. (i) *Assume that we start with 15 stones. Then, by Definition 12,* $(15) \in \mathcal{P}$*. Since* $\lfloor \frac{15}{2} \rfloor = 7$*, one can remove seven or fewer stones. Then, one moves to* (n) *such that* $8 \leq n \leq 14$*. Therefore, a player always moves to a position that is not in* $\mathcal{P}$ *(Figure 5).*
(ii) *Assume that we start with 11 stones; then* $(11) \notin \mathcal{P}$*. Since* $\lfloor \frac{11}{2} \rfloor = 5$*, one can remove five or fewer stones. Thus, one can move to* $(7) \in \mathcal{P}$*. Similarly, one can move to* $(3) \in \mathcal{P}$ *from* (6) *(Figure 6).*

Lemma 14.

(i) Starting with $(x) \in \mathcal{P}$*, one always moves to a position* $(u) \in \mathcal{N}$*.*

(ii) Starting with $(x) \in \mathcal{N}$*, one can always move to a position* $(u) \in \mathcal{P}$*.*

Proof. (i) Assume that one starts with $(x) \in \mathcal{P}$. By Definition 12, $x = 2^n - 1$ for some $n \in \mathbb{Z}_{\geq 0}$. If $n = 0$, then $(x) = (0)$, and one cannot move any more. Next, we assume that $n \geq 1$. From this position, any move leads to a position $(x - u)$ such that $u \leq \lfloor \frac{x}{2} \rfloor$. Then, $u \leq \lfloor \frac{2^n - 1}{2} \rfloor = 2^{n-1} - 1$, and $2^n - 2 = x - 1 \geq x - u \geq 2^n - 1 - (2^{n-1} - 1) = 2^{n-1}$. Therefore, $(x - u) \notin \mathcal{P}$.
(ii) Assume that one starts with $(x) \in \mathcal{N}$. By Definition 12, $x \neq 2^n - 1$ for any $n \in \mathbb{Z}$. Then there exists $n \in \mathbb{Z}$ such that

$$2^n - 1 < x < 2^{n+1} - 1 \,.$$

Since x is an integer, it follows that $x \leq 2^{n+1} - 2$, and so $\frac{x}{2} \leq 2^n - 1$. Therefore,

$$\frac{x}{2} - \big(x - (2^n - 1)\big) = 2^n - 1 - \frac{x}{2} \geq 0 \,,$$

and hence

$$\left\lfloor \frac{x}{2} \right\rfloor \geq x - (2^n - 1) \,.$$

Since one can remove $\lfloor \frac{x}{2} \rfloor$ or fewer stones, one can move to $(2^n - 1) \in \mathcal{P}$ by removing $x - (2^n - 1)$ stones. □

Theorem 15. *In the game of Definition 9, the sets* $\mathcal{P}$ *and* $\mathcal{N}$ *of Definition 12 are the set of* $\mathcal{P}$*-positions and the set of* $\mathcal{N}$*-positions, respectively.*

Proof. Start the game with a position in $\mathcal{P}$. By Lemma 14 (i) , any option leads to a position $(u) \in \mathcal{N}$. From this position, by Lemma 14 (ii), the opponent can select an appropriate option that leads to a position $(u) \in \mathcal{P}$. Note that any option reduces the number in the coordinate. In this manner, our opponent can always reach a position $(u) \in \mathcal{P}$, and finally, the opponent wins by reaching $(0) \in \mathcal{P}$. Therefore, $\mathcal{P}$ is the set of $\mathcal{P}$-positions.

Now, begin the game with a position $(x) \in \mathcal{N}$. By Lemma 7 (ii), we can select an appropriate option that leads to a position (p) in $\mathcal{P}$. From (p), any option our opponent selects leads to a position in $\mathcal{N}$. In this manner, we win the game by reaching (0). Therefore, $\mathcal{N}$ is the set of $\mathcal{N}$-positions. □

3.2 Restricted Nim with two types of stones

In this subsection, we study new games proposed by Shoei Takahashi. In the traditional game of Nim, we use only one type of stone. Here, we have two kinds of stones with different weights.

Definition 16. Suppose that there is a pile of stones, and two players take turns removing stones from the pile. There are two types of stones. We call a stone Type 1 when its weight is 1 and Type 2 when its weight is 2. A player can remove stones whose total weight is less than or equal to the total weight of the remaining stones in the pile. Hence, when there are x stones of Type 1 and y stones of Type 2, a player can remove stones whose total weight is less than or equal to $\lfloor \frac{x+2y}{2} \rfloor$, where $\lfloor \ \rfloor$ is the floor function.

We denote a position in this game by (x, y), where x and y are the numbers of Type 1 stones and Type 2 stones, respectively.

We define another game.

Definition 17. The rule of the game is the same as that of the game in Definition 16, except that a player can remove stones whose total weight is less than or equal to $\lceil \frac{x+2y}{2} \rceil$, when there are x stones of Type 1 and y stones of Type 2. Here, $\lceil \ \rceil$ is the ceiling function.

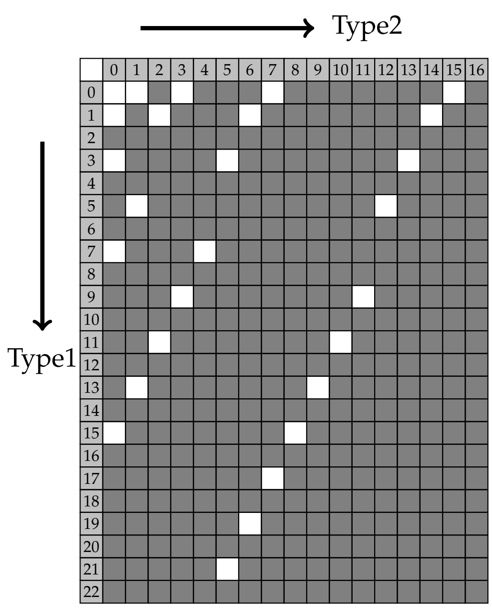

Figure 7: $\mathcal{P}$-positions are indicated by white rectangles

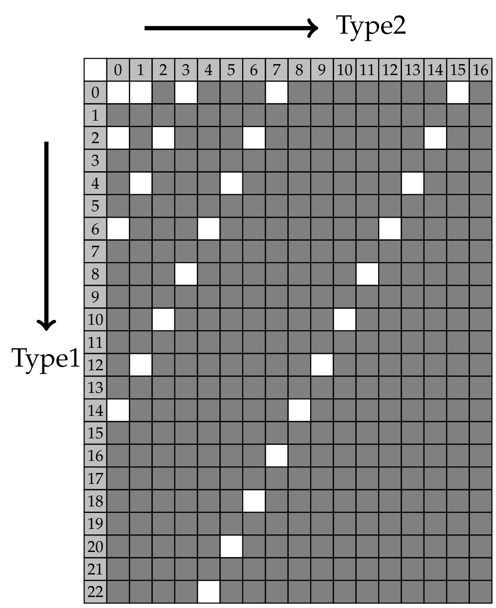

Figure 8: $\mathcal{P}$-positions are indicated by white rectangles

By calculating the $\mathcal{P}$-positions of games of Definitions 16 and 17, we obtain the graphs shown in Figures 7 and 8. The set of $\mathcal{P}$-positions in Figure 8 is very simple; hence, we study the game of Definition 17 in this section.

An article regarding the game of Definition 16 has already been submitted to a journal; see [9]. The graph in Figure 7 is not complicated; however, the proof for this game is very complicated.

Remark 18. After formulating a new game, the next step is to develop a computer program to calculate the $\mathcal{P}$-positions. Then, mathematical formulas describing the data can be found. Mathematicians usually discover formulas by trial and error; however, the present authors use AI to find these formulas, as shown in [12] and [13].

Definition 19. For $x, y \in \mathbb{Z}_{\geq 0}$, the set of all the positions that can be reached from position (x, y) is defined as *move*(x, y). Then, we have

$$\textit{move}(x,y) = \left\{(x-t, y-u) : t, u \in \mathbb{Z}_{\geq 0}, \text{ and } 1 \leq t + 2u \leq \left\lceil \frac{x+2y}{2} \right\rceil \right\}.$$

Definition 20. For $n \in \mathbb{Z}_{\geq 0}$, let

$$\mathcal{P}_n = \left\{(2i-2, 2^n - i) : i \in \mathbb{N} \text{ and } i \leq 2^n\right\} \qquad \text{and} \qquad \mathcal{P} = \bigcup_{n=0}^{\infty} \mathcal{P}_n\,.$$

Remark 21. Note that $\mathcal{P}_0 = \{(0, 0)\}$ in Definition 20.

Next, we prove that if one starts with $(x, y) \in \mathcal{P}$, then one always moves to a position $(u, v) \notin \mathcal{P}$, i.e., move$(x, y) \cap \mathcal{P} = \emptyset$.

Lemma 22. *If we start with a position $(x, y) \in \mathcal{P}$, then move$(x, y) \cap \mathcal{P} = \emptyset$.*

Proof. Suppose that we start with a position $(x, y) = (2i - 2, 2^n - i) \in \mathcal{P}_n$. Then the total weight of the stones is

$$(2i-2) + 2(2^n - i) = 2^{n+1} - 2\,, \tag{1}$$

so, by the definition of the game, we can remove stones whose total weight is equal to or less than

$$\left\lceil \frac{2^{n+1}-2}{2} \right\rceil = 2^n - 1\,. \tag{2}$$

We prove the lemma by contradiction. Assume that $(u, v) \in$ move(x, y) for some position $(u, v) = (2j - 2, 2^t - j) \in \mathcal{P}_t$ where $t, j \in \mathbb{N}$ with $1 \leq j \leq 2^t$. Then the total weight of the stones is

$$2^{t+1} - 2\,. \tag{3}$$

Hence, by (1) and (3),

$$t \leq n - 1\,. \tag{4}$$

If we move from (x, y) to (u, v), then by (1), (3) and (4), we have to remove stones whose total weight is

$$2^{n+1} - 2 - (2^{t+1} - 2) \geq 2^{n+1} - 2^n \geq 2^n\,. \tag{5}$$

However, (5) contradicts (2). Therefore, for $t \leq n - 1$, move$(x, y) \cap \mathcal{P}_t = \emptyset$. □

Next, we prove that if one starts with $(x, y) \notin \mathcal{P}$, then one can always moves to a position $(u, v) \in \mathcal{P}$, i.e., move$(x, y) \cap \mathcal{P} \neq \emptyset$. This is stated in Lemma 24 below. First however, we study the method of proof by some examples.

Example 23. (a) *Suppose that we start with a position $(8,9) \notin \mathcal{P}$ in Figure 9. Then, we can remove stones whose total weight is $\lceil \frac{8+2\times 9}{2} \rceil = 13$, and we can move to $(8,3) \in \mathcal{P}$ by removing stones whose total weight is 12. From $(12,8)$ and $(14,3)$, we can move to a position in $\mathcal{P}$ by a similar method. These moves are represented by green rectangles and arrows.*

(b) *In Figure 9, we assume that we start with a position $(18,5) \notin \mathcal{P}$. Then, we can remove stones whose total weight is $\lceil \frac{18+2\times 5}{2} \rceil = 14$, and we can move to $(4,5) \in \mathcal{P}$ by removing stones whose total weight is 14. From $(16,2)$ and $(14,7)$, we can move to a position in $\mathcal{P}$ by a similar method. These moves are represented by blue rectangles and arrows.*

(c) *In Figure 10, we assume that we start with a position $(13,8) \notin \mathcal{P}$. Then, we can remove stones whose total weight is $\lceil \frac{13+2\times 8}{2} \rceil = 15$, and we can move to $(12,1) \in \mathcal{P}$ by removing stones whose total weight is 15. From $(9,9)$ and $(15,3)$, we can move to a position in $\mathcal{P}$ by a similar method. These moves are represented by green rectangles and arrows.*

(d) *In Figure 10, we assume that we start with a position $(12,7) \notin \mathcal{P}$. Then, we can remove stones whose total weight is $\lceil \frac{12+2\times 7}{2} \rceil = 13$, and we can move to $(0,7) \in \mathcal{P}$ by removing stones whose total weight is 12. From $(17,2)$ and $(15,5)$, we can move to a position in $\mathcal{P}$ by a similar method. These moves are represented by blue rectangles and arrows.*

(e) *In Figure 10, we assume that we start with a position $(3,6) \notin \mathcal{P}$. Then, we can remove stones whose total weight is $\lceil \frac{3+2\times 6}{2} \rceil = 7$, and we can move to $(2,6) \in \mathcal{P}$ by removing stones whose total weight is 1. From $(7,12)$, we can move to a position in $\mathcal{P}$ by a similar method. These moves are represented by red rectangles and arrows.*

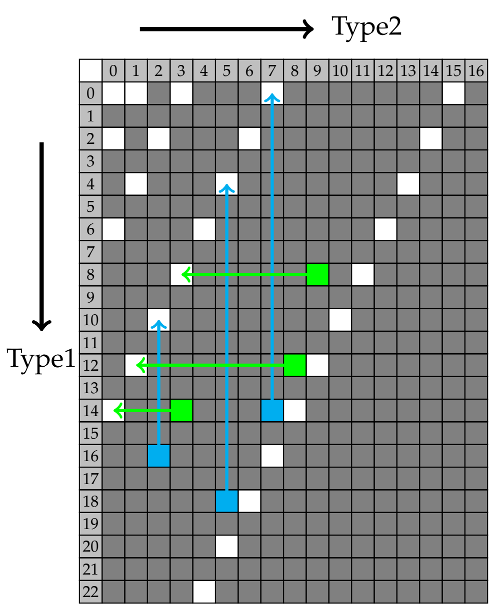

Figure 9: How to move to a $\mathcal{P}$-position

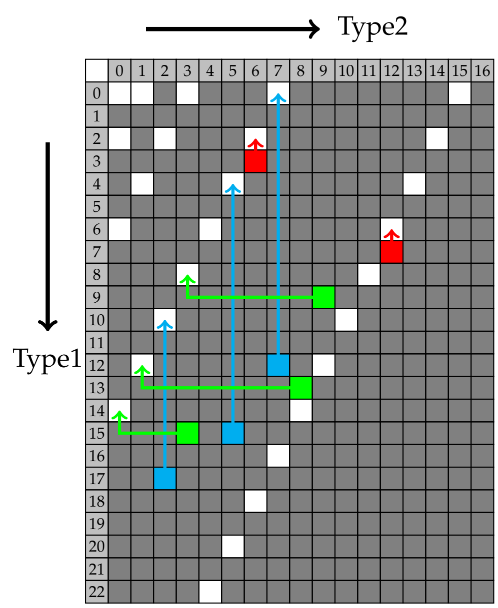

Figure 10: How to move to a $\mathcal{P}$-position

Lemma 24. *If we start with a position $(x,y) \notin \mathcal{P}$, then move$(x,y) \cap \mathcal{P} \neq \emptyset$.*

Proof. Suppose that $(x, y) \notin \mathcal{P}$.
(i) If x is even, then there exists $i \in \mathbb{N}$ such that $(x, y) = (2i - 2, y)$. Since $(x, y) \notin \mathcal{P}$ and by Definition 20, $y \neq 2^t - i$ for any $t \in \mathbb{Z}_{\geq 0}$, and there exists $n \in \mathbb{Z}_{\geq 0}$ such that

$$2^n - i < y < 2^{n+1} - i\,. \tag{6}$$

Then, the total weight of the stones is $2y + 2i - 2$, and the total weight of stones that can be removed is

$$y + i - 1\,. \tag{7}$$

(i.1) Suppose that $2^n - i > 0$. Note that examples of this case are in (a) of Example 23. In these examples, we move from green rectangles that are positions not in $\mathcal{P}$ to white rectangles that are positions in $\mathcal{P}$ along green arrows in Figure 9. Note that to move to $(2i - 2, 2^n - i)$ from $(2i - 2, y)$, we need to remove stones with total weight

$$2(y - (2^n - i)) = 2y + 2i - 2^{n+1}\,. \tag{8}$$

By (6),

$$y + i - 1 - (2y + 2i - 2^{n+1}) = 2^{n+1} - i - y - 1 \geq 0\,. \tag{9}$$

Therefore, by (7), (8) and (9), we can move to $(2i - 2, 2^n - i) \in \mathcal{P}$ from $(2i - 2, y)$.
(i.2) Suppose that

$$2^n - i < 0 \leq y < 2^{n+1} - i\,. \tag{10}$$

Note that examples of this case are in (b) of Example 23. In these examples, we move from blue rectangles that are positions not in $\mathcal{P}$ to white rectangles that are positions in $\mathcal{P}$ along blue arrows in Figure 9. By (10), $2^n < i$; hence,

$$2^{n+1} - i \leq 2^{n+1} - 2^n - 1 = 2^n - 1\,. \tag{11}$$

By (10) and (11),

$$2^n - i < 0 \leq y < 2^n - 1\,,$$

so there exists $j \in \mathbb{N}$ such that $1 < j < i$ and $y = 2^n - j$. Then, by (7), one can, at the position $(2i - 2, y)$, remove stones whose total weight is

$$2^n - j + i - 1\,. \tag{12}$$

By (10), we have

$$y = 2^n - j < 2^{n+1} - i\,. \tag{13}$$

By (13), $i - j < 2^n$, so

$$2(i - j) \leq 2^n + i - j - 1\,. \tag{14}$$

Therefore, by (12) and (14), one can move to the position $(2j-2, 2^n-j)$ from the position $(2i - 2, 2^n - j)$ by removing stones whose total weight is $2(i - j)$.
(ii) If x is odd, then there exists $i \in \mathbb{N}$ such that $(x, y) = (2i - 1, y)$. Then, the total weight of the stones is $2y + 2i - 1$, and the total weight of stones that can be removed is

$$\left\lceil \frac{2y + 2i - 1}{2} \right\rceil = y + i\,. \tag{15}$$

$(\mathrm{ii}.1)$ Suppose that $y = 2^n - i$ for some $n \in \mathbb{Z}_{\geq 0}$. Note that examples of this case are in (e) of Example 23. In these examples, we move from red rectangles that are positions not in $\mathcal{P}$ to white rectangles that are positions in $\mathcal{P}$ along red arrows in Figure 10. By (15), we can move to $(2i-2, y) \in \mathcal{P}$ by removing one stone from the first pile.
$(\mathrm{ii}.2)$ Suppose that

$$y \neq 2^t - i \tag{16}$$

for any $t \in \mathbb{Z}_{\geq 0}$. Then, there exists $n \in \mathbb{Z}_{\geq 0}$ such that

$$2^n - i < y < 2^{n+1} - i\,. \tag{17}$$

$(\mathrm{ii}.2.1)$ Suppose that $2^n - i \geq 0$. Note that examples of this case are in (c) of Example 23. In these examples, we move from green rectangles that are positions not in $\mathcal{P}$ to white rectangles that are positions in $\mathcal{P}$ along green arrows in Figure 10. To move to position $(2i-2, 2^n - i)$ from $(2i-1, y)$, we need to remove stones whose total weight is

$$2(y - (2^n - i)) + 1 = 2(y + i - 2^n) + 1 = 2y + 2i + 1 - 2^{n+1}\,. \tag{18}$$

By (17),

$$y + i - (2y + 2i + 1 - 2^{n+1}) = 2^{n+1} - i - y - 1 \geq 0\,. \tag{19}$$

Therefore, by (15), (18) and (19), we can move to $(2i-2, 2^n - i)$ from $(2i-1, y)$.
$(\mathrm{ii}, 2.2)$ Suppose that

$$2^n - i < 0 \leq y < 2^{n+1} - i\,. \tag{20}$$

Note that examples of this case are in (d) of Example 23. In these examples, we move from blue rectangles that are positions not in $\mathcal{P}$ to white rectangles that are positions in $\mathcal{P}$ along blue arrows in Figure 10. By (20), $2^n < i$; hence, $2^{n+1} - i \leq 2^{n+1} - 2^n - 1 = 2^n - 1$. Therefore, by (20),

$$2^n - i < 0 \leq y < 2^n - 1\,. \tag{21}$$

Then, there exists $j \in \mathbb{N}$, such that $1 < j < i$ and $y = 2^n - j$. By (15), at the position $(2i-1, y)$, one can remove stones whose total weight is

$$y + i = 2^n - j + i\,. \tag{22}$$

By (20), $y = 2^n - j < 2^{n+1} - i$; hence,

$$i - j < 2^n\,. \tag{23}$$

By (23), $2(i-j) \leq 2^n + i - j - 1$, so by (22), one can move from $(2i-1, y) = (2i-1, 2^n - j)$ to position $(2j-2, 2^n - j)$ by removing stones whose total weight is $2(i-j)+1$. □

Theorem 25. *$\mathcal{P}$ is the set of $\mathcal{P}$-positions of the game of Definition 17.*

Proof. The method of this proof is similar to the proof used for Theorem 15. Here, we use Definition 20, Lemma 22 and Lemma 24. □

3.3 When the restriction depends on the turn

Traditionally, limits on the number of stones to be removed depend on the number of stones in the pile. In the new game introduced in this subsection, the player may at the k-th turn remove at least one stone and at most $f(k)$ stones, here $f(k)$ is a function of k.

Definition 26. Let $f(k)$ be a function whose values are natural numbers. Suppose there is a pile of stones, and two players take turns removing stones from the pile. In the k-th turn, the player is allowed to remove at least one stone and at most $f(k)$ stones. The player who removes the last stone is the winner. When a player removes stones in the k-th turn and the number of stones is n, we denote (n, k) as the position of the game.

Definition 27. For $u, k \in \mathbb{N}$, the set of all the positions that can be reached from position (u, k) is defined as *move*(u, k). Define move$(0, k) = \emptyset$ and, for $u, k \in \mathbb{N}$, define

$$\text{move}(u, k) = \left\{(u - t, k + 1) : t \in \mathbb{N} \text{ and } t \leq \min(u, f(k))\right\}.$$

When no stones are left, no more stones can be removed, so move$(0, k) = \emptyset$. When we move from (u, k), the number of stones to be removed is at most $f(k)$. Since we cannot remove more stones than there are stones, we can remove at least one and at most $\min(u, f(k))$ stones, and its destination is the position $(u - t, k + 1)$ with $t \leq \min(u, f(k))$. Note that $k + 1$ means the $k + 1$-th turn of the game.

3.4 When $f(k) = mk$ for some natural number m

Here, we study the case where $f(k) = mk$ for a natural number m.

Example 28. *Let $m = 1$. By computer calculations, we have the set of $\mathcal{P}$-positions for (x, k) where $x = 0, 1, \ldots, 30$ and $k = 1, 2, \ldots, 15$, as shown in Figure 11.*

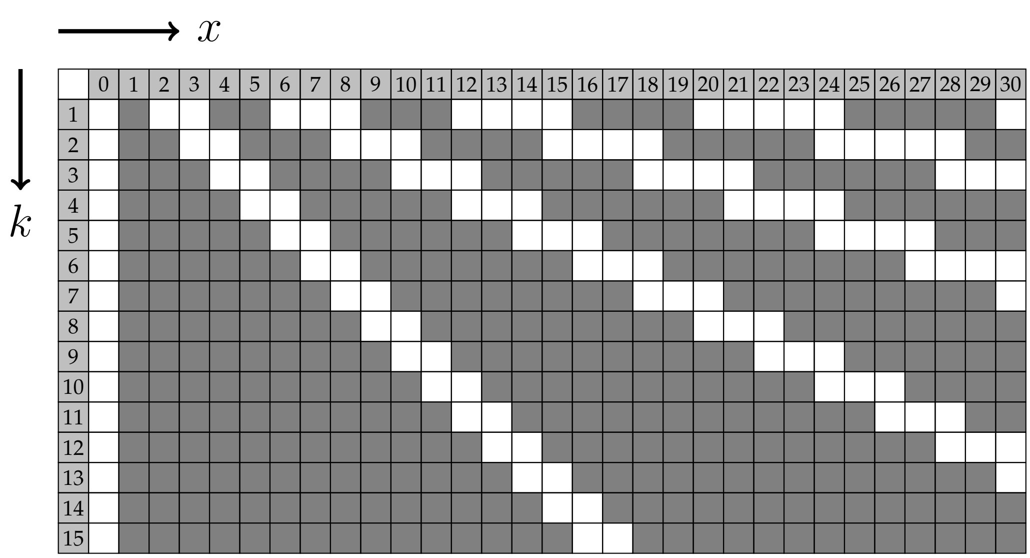

Figure 11: $\mathcal{P}$-positions, indicated by white rectangles.

Definition 29. For $k, n \in \mathbb{Z}_{\geq 0}$, define

$$\mathcal{P}^m_{k,n} = \{(n(mn + m(k-1) + 1) + i, k) : i \in \mathbb{Z}_{\geq 0} \text{ and } i \leq mn\}.$$

Also, define

$$\mathcal{P}^m_k = \bigcup_{n \in \mathbb{Z}_{\geq 0}} \mathcal{P}^m_{k,n} \quad \text{and} \quad \mathcal{P}^m = \bigcup_{k \in \mathbb{N}} \mathcal{P}^m_k.$$

Remark 30. Note that $\mathcal{P}^m_{k,0} = \{(0, k)\}$.

Lemma 31. *For* $(u, k) \in \mathcal{P}^m_{k,n}$ *and* $(v, k) \in \mathcal{P}^m_{k,n+1}$, *we have*

$$u < v. \tag{24}$$

Proof. The element of $\mathcal{P}^m_{k,n+1}$ with the smallest first coordinate is $((n+1)(m(n+1) + m(k-1) + 1), k)$, and the element of $\mathcal{P}^m_{k,n}$ with the largest first coordinate is $(n(mn + m(k-1) + 1) + mn, k)$. Since

$$(n+1)(m(n+1) + m(k-1) + 1) - (n(mn + m(k-1) + 1) + mn) = mn + km + 1 > 0,$$

Definition 29 implies (24). $\square$

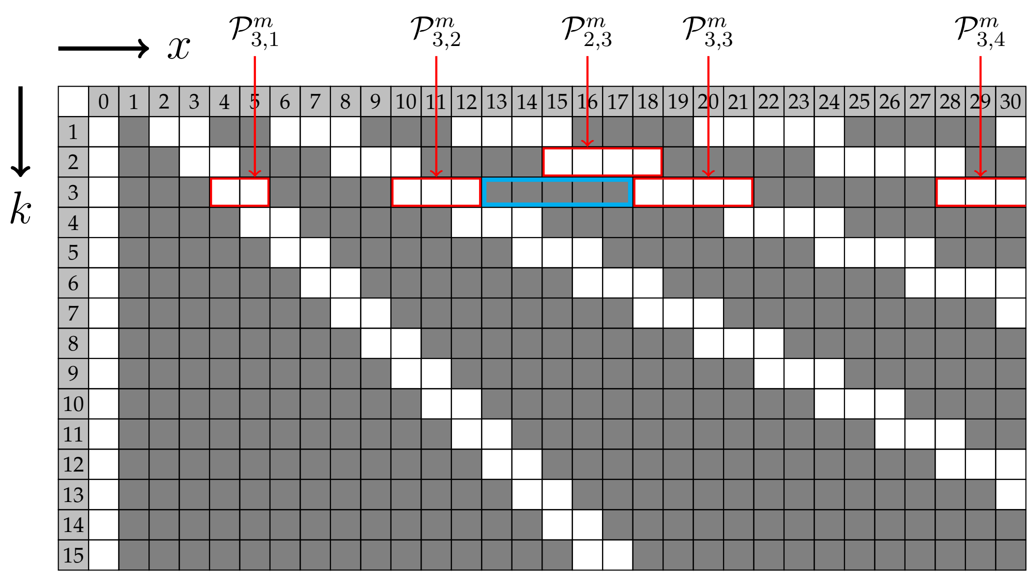

Figure 12: The method used to prove Lemma 33

Our next aim is to prove that the set $\mathcal{P}^m$ in Definition 29 is the set of $\mathcal{P}$-positions. For this purpose, we need Lemmas 31, 33 and 34.

We present and prove Lemma 33 below, but the proof is somewhat difficult to understand, so we first present Example 32 to make the proof easier to understand. Since white rectangles are $\mathcal{P}$-positions, you move to gray rectangles if you start with a white rectangle.

Example 32. *In Figure 12, by starting with a position $(x, 2) \in \mathcal{P}^m_{2,3}$, one moves to the position $(u, 3)$ in the blue rectangle. To prove this, we prove that u is larger than the first coordinate of elements of $\mathcal{P}^m_{3,2}$, and smaller than the first coordinate of elements of $\mathcal{P}^m_{3,3}$. Note that the set $\mathcal{P}^m_{3,1}$ is on the left of the set $\mathcal{P}^m_{3,2}$, and the set $\mathcal{P}^m_{3,4}$ is on the right of the set $\mathcal{P}^m_{3,3}$.*

Lemma 33. *If we start with a position $(x, k) \in \mathcal{P}^m_k$, then move$(x, k) \cap \mathcal{P}^m_{k+1} = \emptyset$.*

Proof. Suppose that $(x, k) \in \mathcal{P}^m_k$ and $(u, k+1) \in \text{move}(x, k)$. Then there exists $n \in \mathbb{Z}_{\geq 0}$ such that $(x, k) \in \mathcal{P}^m_{k,n}$. If $(x, k) \in \mathcal{P}^m_{k,0} = \{(0, k)\}$, then one cannot move to any position.

Next, we assume that $n \geq 1$. By Definition 29

$$n(mn + m(k-1) + 1) \leq x \leq n(mn + m(k-1) + 1) + mn\,. \tag{25}$$

By this game's definition, we can on the k-th turn remove t stones, where $1 \leq t \leq mk$. Hence,

$$x - mk \leq u \leq x - 1\,. \tag{26}$$

By (25) and (26), we have

$$n(mn + m(k-1) + 1) - mk \leq u \leq n(mn + m(k-1) + 1) + mn - 1\,. \tag{27}$$

Since

$$n(mn + m(k-1) + 1) - mk - ((n-1)(m(n-1) + mk + 1) + m(n-1)) = 1 > 0\,,$$

we have

$$(n-1)(m(n-1) + mk + 1) + m(n-1) < n(mn + m(k-1) + 1) - mk\,. \tag{28}$$

Since

$$\mathcal{P}^m_{k+1,n-1} = \{((n-1)(m(n-1) + mk + 1) + i, k+1) : i \in \mathbb{Z}_{\geq 0} \text{ and } i \leq m(n-1)\}\,,$$

the left term of (28) is the maximum value of the first coordinate of the elements in $\mathcal{P}^m_{k+1,n-1}$. By (27), the right term of (28) is the minimum value of u. Therefore, we have

$$v < u \tag{29}$$

for any $(v, k+1) \in P^m_{k+1,n-1}$. By (29) and Lemma 31, we have

$$v < u$$

for any $(v, k+1) \in P^m_{k+1,t}$ such that $t \leq n-1$. Therefore, we have

$$(u, k+1) \notin P^m_{k+1,t} \tag{30}$$

for any $t \leq n-1$. Note that

$$n(mn + m(k-1) + 1) + mn - 1 = n(mn + mk + 1) - 1 < n(mn + mk + 1)$$

and

$$\mathcal{P}_{k+1,n}^m = \{(n(mn+mk+1)+i, k+1) : i \in \mathbb{Z}_{\geq 0} \text{ and } i \leq mn\}\,.$$

By (27), the maximum value of u is smaller than the minimum value of the first coordinate of the elements in $\mathcal{P}_{k+1,n}^m$. Therefore,

$$u < v \tag{31}$$

for any $(v, k+1) \in P_{k+1,n}^m$. By (31) and Lemma 31, we have

$$u < v$$

for any $(v, k+1) \in P_{k+1,t}^m$ such that $t \geq n$. Therefore, we have

$$(u, k+1) \notin P_{k+1,t}^m \tag{32}$$

for any $t \geq n$, and (30) and (32) conclude the proof. □

Lemma 34. *If we start with a position $(x, k) \notin \mathcal{P}_k^m$, then move$(x, k) \cap \mathcal{P}_{k+1}^m \neq \emptyset$.*

Proof. Suppose that we start with a position $(x, k) \notin \mathcal{P}_k^m$. Then there exists $n \in \mathbb{N}$ such that

$$u < x < v$$

for any $(u, k) \in \mathcal{P}_{k,n-1}^m$ and $(v, k) \in \mathcal{P}_{k,n}^m$. Then, by Definition 29,

$$x_1 \leq x \leq x_2\,,$$

where

$$\begin{aligned} x_1 &= (n-1)(m(n-1)+m(k-1)+1)+m(n-1)+1 \\ &= (n-1)(m(n-1)+mk+1)+1 \end{aligned} \tag{33}$$

and

$$x_2 = n(mn+m(k-1)+1)-1\,. \tag{34}$$

See points x_1 and x_2 in Figure 13. We have two cases: (i) and (ii).
(i) Suppose that $n = 1$. Then

$$1 = x_1 \leq x \leq x_2 = mk\,.$$

Since we can remove mk stones from the pile, we can move to $(0, k+1) \in \mathcal{P}_{k+1,0}^m$.
(ii) Suppose that $n \geq 2$ (Figures 13 and 14). Let

$$t_1 = (n-1)(m(n-1)+mk+1) \tag{35}$$

and

$$\begin{aligned} t_2 &= (n-1)(m(n-1)+mk+1)+m(n-1) \\ &= n(mn+m(k-1)+1)-mk-1\,. \end{aligned} \tag{36}$$

Then, we have $\mathcal{P}^m_{k+1,n-1} = \{(w, k+1) : t_1 \le w \le t_2\}$; see points t_1 and t_2 in Figure 13. By (33) and (35),

$$x_1 - t_1 = 1\,. \tag{37}$$

By (34) and (36),

$$x_2 - t_2 = mk\,. \tag{38}$$

By (33) and (36),

$$t_2 - x_1 = mn - m - 1\,. \tag{39}$$

We prove that we can move to $\mathcal{P}^m_{k+1,n-1}$. In Figure 14, (x, k) is in a blue rectangle, and we can move to a green rectangle, which is $\mathcal{P}^m_{k+1,n-1}$.

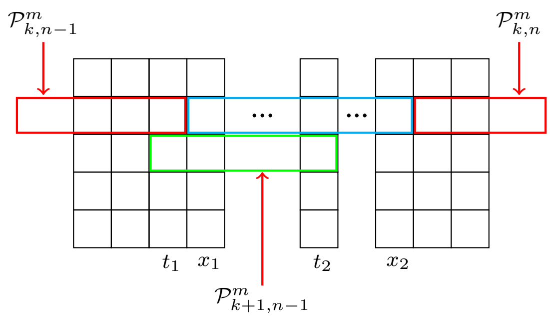

Figure 13: The positions of $\mathcal{P}^m_{k,n-1}$, $\mathcal{P}^m_{k,n}$ and $\mathcal{P}^m_{k+1,n-1}$

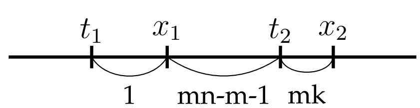

Figure 14: The positions of t_1, x_1, t_2 and x_2

We have two cases: $(ii.1)$ and $(ii.2)$ (Figure 14).

$(ii.1)$ Suppose that we have

$$t_2 + 1 \le x \le x_2. \tag{40}$$

Since we can remove mk or fewer stones, by (38), we can move to $(t_2, k+1) \in \mathcal{P}^m_{k+1,n-1}$ from (x, k).

$(ii.2)$ Suppose that we have

$$x_1 \le x \le t_2\,.$$

By (37) and (39), we can move to $(x-1, k+1) \in \mathcal{P}^m_{k+1,n-1}$. □

Theorem 35. *$\mathcal{P}^m$ is the set of $\mathcal{P}$-positions of the game of Definition 26.*

Proof. Using a method that is similar to the one used to prove Theorems 8 and 15, we prove this theorem using Definition 29, Lemma 33 and Lemma 34. □

3.5 Cases for other types of functions.

The authors have calculated the sets of $\mathcal{P}$-positions for various types of functions $f(k)$ using computers.

If we compare the graphs in Figures 11, 15 and 16, then they appear similar when $f(k)$ is a polynomial of k.

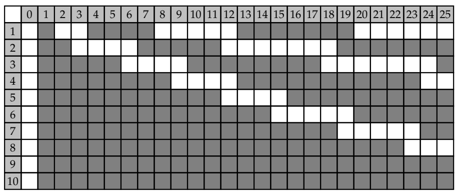

Figure 15: $f(k) = k^{1.5}$

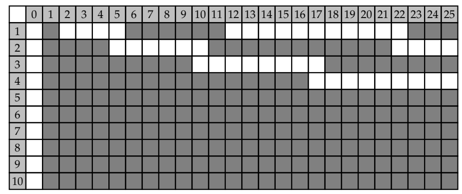

Figure 16: $f(k) = k^2$

When $f(k) = \lfloor \log_2 k \rfloor + 1$ or $f(k) = \lfloor \log_{10} k \rfloor + 1$, interesting graphs are obtained (Figures 17 and 18). When $f(k)$ is the k-th number of the Fibonacci sequence or Tribonacci sequence, interesting graphs are obtained as well (Figures 19 and 20). The authors have not discovered any formula that describes these sets of $\mathcal{P}$-positions.

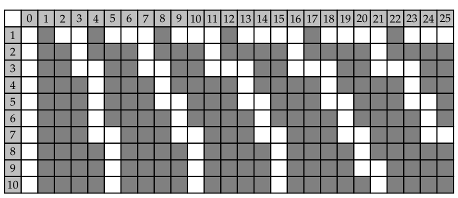

Figure 17: $f(k) = \lfloor \log_2 k \rfloor + 1$

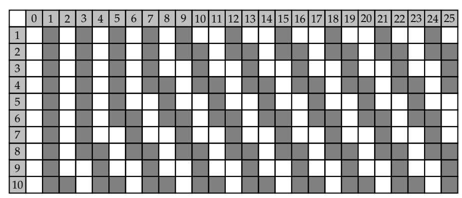

Figure 18: $f(k) = \lfloor \log_{10} k \rfloor + 1$

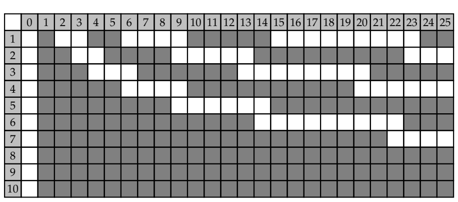

Figure 19: Fibonacci sequence where $f(1) = 1$, $f(2) = 2$

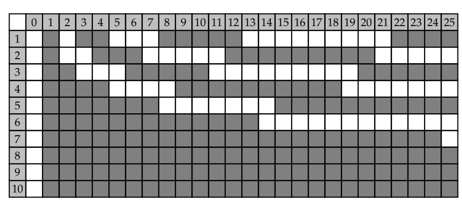

Figure 20: Tribonacci sequence where $f(1) = 1$, $f(2) = 1$, $f(3) = 2$

4 Prospect for further research

When the conditions of the games studied in this article are changed, one can study new games.

The following Mathematica program is for the game studied in Subsection 3.2. Calculations can be implemented without the software Mathematica by pasting this program into Wolfram Playground [14]. This program can also be found on the GitHub site [15].

In Subsection 3.2, we used stones of weights one and two. In this program, the weights of stones are $hh = 1$ and $kk = 2$. If we let $hh = 3$ and $kk = 6$, then one can calculate the $\mathcal{P}$-positions of the game with stones weighing three and six, respectively.

A Python program for this game can be found on the GitHub site [16].

```
Clear[al,move,Mex,Gr,gote,kk,ss,pposition];
hh=1;kk=2;ss=30;al=Flatten[Table[{a,b},{a,0,ss},{b,0,ss}],1];
move[z_]:=Block[{p},p=z;Select[Flatten[Table[{p[[1]]-k,p[[2]]-h},
  {k,0,p[[1]]},{h,0,p[[2]]}],1],
  0<hh*(p[[1]]-#[[1]])+kk(p[[2]]-#[[2]])
  <=Ceiling[(hh*p[[1]]+kk*p[[2]])/2]&]];
Mex[L_]:=Min[Complement[Range[0,Length[L]],L]];
Gr[pos_]:=Gr[pos]=Mex[Map[Gr,move[pos]]];
pposition=Select[al,Gr[#]==0&];
ff[x_]:=If[{x[[1]],x[[2]]}=={-1,-1},"",
  Which[x[[2]]==-1,x[[1]],x[[1]]==-1,x[[2]]]];
bl=Flatten[Table[{n,m},{n,2,ss+2},{m,1,ss+2}],1];
aa1=Table[{n,1},{n,2,ss+2}];
aa2=Table[{1,n},{n,2,ss+2}];
Grid[Table[ff[{n,m}],{n,-1,ss},{m,-1,ss}],Frame->All,
  Background->{None,None,
    Join[Table[bl[[s]]->GrayLevel[0.7],{s,1,Length[bl]}],
    Table[aa1[[s]]->GrayLevel[0.9],{s,1,Length[aa1]}],
    Table[aa2[[s]]->GrayLevel[0.9],{s,1,Length[aa2]}],
    Table[{pposition[[s,1]]+2,pposition[[s,2]]+2}
    ->White,{s,1,Length[pposition]}]]}]
```

The following Mathematica program is for the game studied in Subsections 3.3 and 3.4. This program (`takahasi.txt`) can also be found on the GitHub site [17]. A Python program for this game can be found on the GitHub site [18].

```
Clear[al,move,Gr,pposition,f];ss=40;
al=Flatten[Table[{a,b},{a,0,ss},{b,1,ss}],1];
Table[f[n]=n,{n,0,2ss}];
move[z_]:=Block[{p},p=z;If[p[[1]]==0,{},
  Table[{p[[1]]-k,p[[2]]+1},{k,1,Min[p[[1]],f[p[[2]]]]}]]];
Mex[L_]:=Min[Complement[Range[0,Length[L]],L]];
Gr[pos_]:=Gr[pos]=Mex[Map[Gr,move[pos]]];
pposition0=Select[al,Gr[#]==0&];
pposition=Map[{#[[2]],#[[1]]}&,pposition0];
ff[x_]:=If[{x[[1]],x[[2]]}=={0,-1},"",
  Which[x[[2]]==-1,x[[1]],x[[1]]==0,x[[2]],1==1,]];
bl=Flatten[Table[{n,m},{n,2,ss+2},{m,1,ss+2}],1];
aa1=Table[{n,1},{n,2,30}];
aa2=Table[{1,n},{n,2,30+2}];
Grid[Table[ff[{n,m}],{n,0,11},{m,-1,30}],Frame->All,
  Background->{None,None,
    Join[Table[bl[[s]]->GrayLevel[0.7],{s,1,Length[bl]}],
    Table[aa1[[s]]->GrayLevel[0.9],{s,1,Length[aa1]}],
    Table[aa2[[s]]->GrayLevel[0.9],{s,1,Length[aa2]}],
    Table[{pposition[[s,1]]+1,pposition[[s,2]]+2}
    ->White,{s,1,Length[pposition]}]]}]
```

We have not studied the Grundy number in this article; however, it is a very important and useful tool for combinatorial game theory. For a detailed explanation of Grundy numbers, see [1].

Acknowledgement

JSPS KAKENHI Grant Number 23H05173 supported this work.

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