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(Yet another) way to prove Pythagoras' Theorem

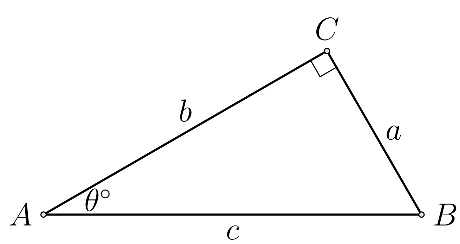
Ajay Kumar K S[1]

1 Introduction

This article presents a new way to prove Pythagoras' Theorem. For centuries, people have used diverse tools such as combinatorics, calculus, geometry, algebra and trigonometry to come up with hundreds of different ways to prove the theorem [1, 2, 3]. The contributors who have shown these new ways have been equally varied and diverse - spanning from ancient mathematicians to contemporary academics, from recreational mathematicians (such as Miss E.A. Coolidge, a remarkable blind girl [1]) to philosophers, from high-school students [4] to a former chief economist of the World Bank [5], and even extending to a former president of the United States [6]. Here, I use some geometry, trigonometry and algebra to prove the theorem. There are few proofs using trigonometry [4, 7, 8]. The reason why I even tried looking for one more way to prove is probably best paraphrased by the character Lisa Simpson from *The Simpsons* - I guess the hunt was more fun than the catch. [9]

2 Pythagoras' Theorem

In the figure below, ΔABC depicts a right-angled triangle, where $\angle ACB = 90°$.



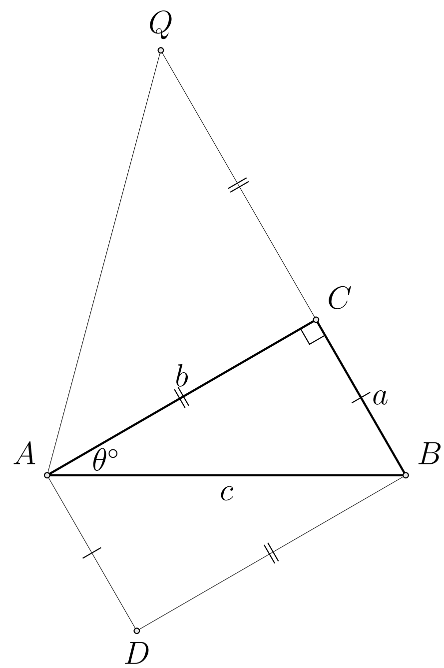
Pythagoras' Theorem can be expressed as follows.

Theorem 1 (Pythagoras' Theorem)**.**
For each right-angled triangle with side lengths a, b, c, where c is longest,

$$a^2 + b^2 = c^2 \,.$$

[1] Ajay Kumar K S is director of pharma solutions at *Strand Life Sciences* (p11ajayks@iima.ac.in).

3 A new proof of Pythagoras' Theorem



From the point C, we construct the triangle ΔACQ with right angle $\angle ACQ = 90^\circ$ and side length $|CQ| = |AC|$. Note that this is an isosceles right-angled triangle with angles $\angle QAC = \angle CQA = 45^\circ$ and $|CQ| = b$. Also note that

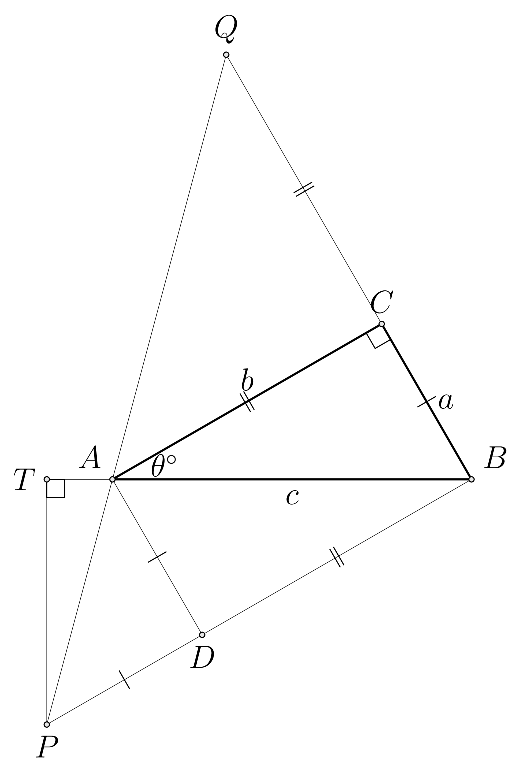
$$\angle QAB = \angle QAC + \angle CAB = (45 + \theta)^\circ \, .$$

We construct another triangle ΔADB with right angle $\angle ADB = 90^\circ$ and side lengths

$$|AD| = |BC| = a$$
$$|DB| = |AC| = b \, .$$

We can see that $\angle BAD = (90 - \theta)^\circ$ and $\angle DBA = \theta^\circ$.

From the point D, we construct the isosceles right-angled triangle ΔDAP with $\angle PDA = 90^\circ$, $\angle DAP = \angle APD = 45^\circ$ and $|AD| = |PD| = a$. Also, from the point P, we draw a line PT that is perpendicular to the line through A and B, giving a triangle ΔATP with $\angle ATP = 90^\circ$ and $\angle PAT = \angle QAB = (45 + \theta)^\circ$.



Note that Q, A and P line on the same straight line and that ΔPBQ is an isosceles right-angled triangle with side lengths $|PB| = |BQ| = a + b$.

In the proof below, we use $\sin$ and $\cos$ functions. We only use their definitions and any of their properties that we can prove without using Pythagoras' Theorem.

By definition,

$$\sin\theta^\circ = \frac{|BC|}{|AB|} = \frac{a}{c} \qquad \text{and} \qquad \cos\theta^\circ = \frac{|AC|}{|AB|} = \frac{b}{c}\,.$$

Also,

$$\sin(90-\theta)^\circ = \sin(\angle ABC) = \frac{|AC|}{|AB|} = \cos\theta^\circ\,,$$

so $\sin 45^\circ = \cos 45^\circ$. Let $\delta = |AT|$; then

$$\delta = |AP|\cos\angle PAT = |AP|\cos(45+\theta)^\circ = \frac{|AD|}{\cos 45^\circ}\cos(45+\theta)^\circ = a\,\frac{\cos(45+\theta)^\circ}{\cos 45^\circ}\,.$$

In the Appendix, we prove that

$$\cos(45+\theta)^\circ = \cos 45^\circ(\cos\theta^\circ - \sin\theta^\circ)\,.$$

This means that

$$\delta = a\,(\cos\theta^\circ - \sin\theta^\circ) = a\left(\frac{b}{c} - \frac{a}{c}\right) = \frac{a}{c}(b-a)\,.$$

Now note that

$$\frac{b}{c} = \cos\theta^\circ = \cos(\angle PBT) = \frac{|BT|}{|BP|} = \frac{|AB| + |AT|}{|PD| + |BD|} = \frac{c+\delta}{a+b}\,,$$

so $\frac{b}{c}(a+b) = c + \delta$. Therefore, $c^2 + \delta c = ab + b^2$, so

$$c^2 = -\delta c + ab + b^2 = -a(b-a) + ab + b^2 = a^2 + b^2\,,$$

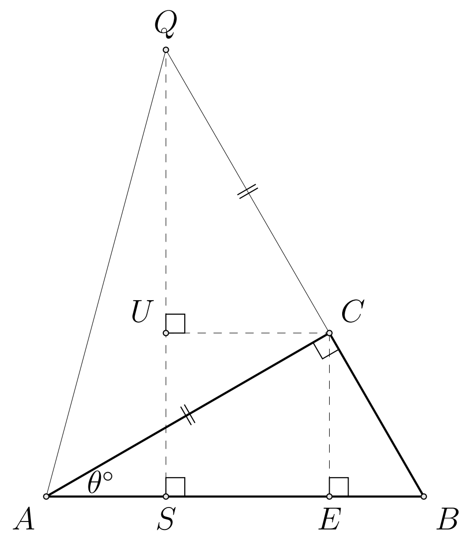
which is what we wanted to prove. $\square$

Appendix: Trigonometry without Pythogoras' Theorem

In this appendix, we prove the formula

$$\cos(45+\theta)^\circ = \cos 45^\circ(\cos\theta^\circ - \sin\theta^\circ)\,.$$

This is a well-known formula [10] but we prove it without using Pythogoras' Theorem, so as to avoid circular logic in our proof above.



From the point Q we draw a line that is perpendicular to AB and meets AB at the point S. From C, we draw a line that is perpendicular to QS and meets QS at the point U. From C, we draw another line, now perpendicular to AB and meeting AB at the point E.

We see that $\angle QAS = (45+\theta)^\circ$ and that $\angle CQU = \angle ACU = \angle CAB = \theta^\circ$, so

$$\cos(45+\theta)^\circ = \frac{|AS|}{|AQ|} = \frac{|AE|-|ES|}{|AQ|} = \frac{|AE|}{|AQ|} - \frac{|UC|}{|AQ|} = \frac{|AE|}{|AC|}\frac{|AC|}{|AQ|} - \frac{|UC|}{|QC|}\frac{|QC|}{|AQ|}\,.$$

Also,

$$\frac{|AE|}{|AC|} = \cos\angle CAE = \cos\theta^\circ \qquad \frac{|AC|}{|AQ|} = \cos\angle QAC = \cos 45^\circ\,;$$
$$\frac{|UC|}{|QC|} = \sin\angle CQU = \sin\theta^\circ \qquad \frac{|QC|}{|AQ|} = \sin\angle QAC = \sin 45^\circ = \cos 45^\circ\,.$$

Therefore,

$$\cos(45+\theta)^\circ = \cos\theta^\circ\cos 45^\circ - \sin\theta^\circ\cos 45^\circ = \cos 45^\circ(\cos\theta^\circ - \sin\theta^\circ)\,,$$

which is what we wanted to show. □

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