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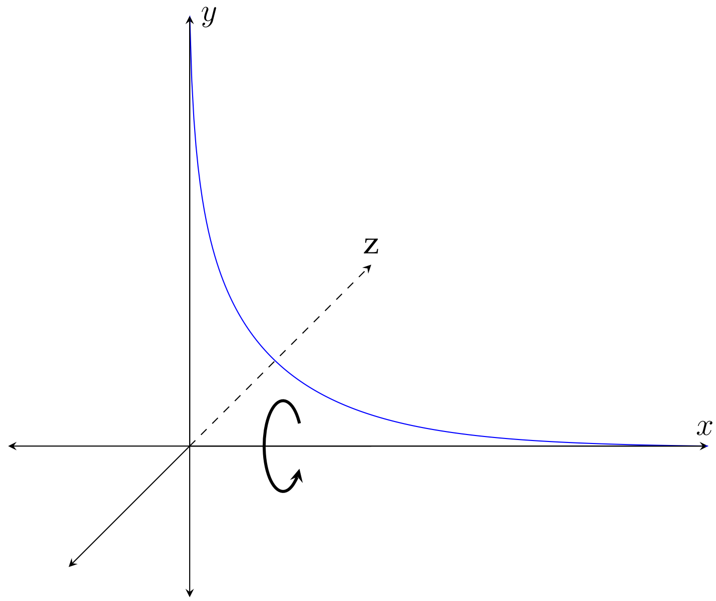
An infinitely bounded region with finite volume

Janelle Powell[1]

1 Introduction

In my previous article [1], we found that the area under the graph of $f(x) = \ln\left(\frac{e^x+1}{e^x-1}\right)$ is equal to $\frac{\pi^2}{4}$. This is particularly interesting, as it poses an example of an infinitely bounded graph with finite area.

In considering this problem, I began to wonder if rotating this graph around the x-axis would yield an infinitely bounded function in the x, y and z directions, but one with finite volume. A visual representation of this is as follows.



2 Using a u-substitution

In my previous article, I found that we could make the substitution

$$\lim_{b\to\infty}\int_0^b \ln\left(\frac{e^t+1}{e^t-1}\right)\,dt = \int_0^1 \frac{\ln\left(\frac{1+x}{1-x}\right)}{x}\,dx\,.$$

[1]Janelle Powell is a first-year student at Bowdoin College, Brunswick, Maine, United States.

The general expression for the volume of a rotated function $f(x)$ is as follows:

$$V = \pi \int_a^b \left(f(x)\right)^2 dx\,.$$

We can apply this to our integration, rotating around the x-axis:

$$V = \pi \int_0^1 \frac{\ln^2\left(\frac{1+x}{1-x}\right)}{x}\, dx\,.$$

We can simplify this integral slightly:

$$\left[\ln\left(\frac{1+x}{1-x}\right)\right]^2 = \left[-\ln\left(\frac{1+x}{1-x}\right)\right]^2 = \left[\ln\left(\frac{1+x}{1-x}\right)^{-1}\right]^2 = \left[\ln\left(\frac{1-x}{1+x}\right)\right]^2 .$$

To simplify further, we can use the u-substitution $u = \frac{1-x}{1+x}$, finding that

$$x = \frac{1-u}{1+u} \qquad \text{and} \qquad dx = \frac{-2}{(1+u)^2}\, du\,.$$

Therefore, we see that

$$V = \pi \int_0^1 \frac{\ln^2\left(\frac{1+x}{1-x}\right)}{x}\, dx = \pi \int_1^0 \frac{\ln^2 u}{\left(\frac{1-u}{1+u}\right)} \frac{-2}{(1+u)^2}\, du = 2\pi \int_0^1 \frac{\ln^2 u}{1-u^2}\, du\,.$$

Since

$$\frac{1}{1-u^2} = \frac{1}{1-u} - \frac{1}{1-u^2}\,,$$

we can split our integral into two:

$$V = 2\pi \int_0^1 \frac{\ln^2 u}{1-u^2}\, du = 2\pi \left(\int_0^1 \frac{\ln^2 u}{1-u}\, du - \int_0^1 \frac{u \ln^2 u}{1-u^2}\, du\right).$$

Splitting our integral offers us more opportunity to use integration techniques that we might not have been able to before. We now have two integrals that we can apply different integration techniques to to try and solve!

3 Applying the substitution $v = u^2$

For our second integral, we can make the substitution $v = u^2$, finding that $\frac{1}{2}dv = u\, du$. Then

$$-\int_0^1 \frac{u \ln^2 u}{1-u^2}\, du = -\frac{1}{2}\int_0^1 \frac{\ln^2 v^{\frac{1}{2}}}{1-v}\, dv = -\frac{1}{2}\int_0^1 \frac{\left(\frac{1}{2}\ln v\right)^2}{1-v}\, dv = -\frac{1}{8}\int_0^1 \frac{\ln^2 v}{1-v}\, dv$$

It's clear that our new integrand matches the integrand of our other integral, just with different variables. Therefore,

$$V = 2\pi \left(\int_0^1 \frac{\ln^2 u}{1-u}\, du - \frac{1}{8}\int_0^1 \frac{\ln^2 v}{1-v}\, dv\right) = \frac{7\pi}{4}\int_0^1 \frac{\ln^2 y}{1-y}\, dy\,.$$

This last expression allows us to apply a function used throughout many disciplines, including calculus and complex analysis: the Gamma Function.

4 Using the Gamma Function as an integration technique

We can rewrite $\frac{1}{1-y}$ as the geometric series $\sum_{n=1}^{\infty} y^{n-1}$. Since this sum converges when $|y| < 1$, which is within our bounds of integration, we can switch our order of integration and summation and rewrite our integral as follows:

$$V = \frac{7\pi}{4}\int_0^1 \frac{\ln^2 y}{1-y}\,dy = \frac{7\pi}{4}\int_0^1 \left(\sum_{n=1}^{\infty} y^{n-1}\right)\ln^2 y\,dy = \frac{7\pi}{4}\sum_{n=1}^{\infty}\int_0^1 y^{n-1}\ln^2 y\,dy\,.$$

This new integrand is reminiscent of our Gamma Function [2] which is defined for all complex numbers z with positive real part as

$$\Gamma(z) = \int_0^{\infty} t^{z-1}e^{-t}\,dt\,.$$

When z is a positive integer, it turns out that

$$\Gamma(z) = (z-1)!\,.$$

The Gamma Function thereby extends the factorial function from positive integers to positive real numbers and, further, to complex numbers with positive real part. Letting $t = -n\ln y$ for some positive integer n, we can take the derivative of t with respect to y and see that $dt = \frac{-n}{y}\,dy$. Note that $y \to 1$ when $t \to 0$ and that $y \to 0$ when $t \to \infty$. Substitution therefore gives us

$$\Gamma(z) = \int_0^{\infty} t^{z-1}e^{-t}\,dt = \int_1^0 (-n\ln y)^{z-1}\,e^{n\ln y}\,\frac{-n}{y}\,dy = (-1)^{z-1}n^z\int_0^1 (\ln y)^{z-1}y^{n-1}\,dy\,.$$

For each positive integer z,

$$(z-1)! = \Gamma(z) = (-1)^{z-1}n^z\int_0^1 (\ln y)^{z-1}y^{n-1}\,dy\,.$$

In particular for $z = 3$, we see that

$$2 = (3-1)! = n^3\int_0^1 (\ln y)^2 y^{n-1}\,dy\,,$$

so

$$\frac{2}{n^3} = \int_0^1 \ln^2 y y^{n-1}\,dy\,,$$

Therefore,

$$V = \frac{7\pi}{4}\sum_{n=1}^{\infty}\int_0^1 y^{n-1}\ln^2 y\,dy = \frac{7\pi}{4}\sum_{n=1}^{\infty}\frac{2!}{n^3} = \frac{7\pi}{2}\sum_{n=1}^{\infty}\frac{1}{n^3}\,.$$

By transforming our integral using the Gamma Function, we have gotten rid of our integral entirely in favour of summation, allowing us to further manipulate it using new techniques.

5 Applying the Riemann Zeta Function

We can identify our sum as $\zeta(3)$, where $\zeta(s)$ is the Riemann Zeta Function of s, defined as follows [3]:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \,.$$

The value of $\zeta(3)$, known as Apéry's Constant [4], is approximately 1.2025. We can therefore approximate our original integral as follows:

$$V = \pi \int_0^1 \frac{\ln^2\left(\frac{1+x}{1-x}\right)}{x}\, dx = \frac{7\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{7\pi}{2}\zeta(3) \approx 13.2173 \,.$$

6 Conclusion

The work that we've done here allows us to conclude what we originally suspected: both the area and rotated volume of the function $f(x) = \ln\left(\frac{e^x+1}{e^x-1}\right)$ are finite. This is fascinating when we consider the implications and applications of infinity in the world of mathematics. Though this is true for this function in particular, it introduces the idea that other functions may have finite area but infinite rotated volume and vice versa.

A famous example of a function with infinite area but finite rotated volume is Gabriel's Horn [5] which rotates the function $f(x) = \frac{1}{x}$ around the x-axis for $x \geq 1$. This example introduces the concept of a square-integrable function [6]. A function $f : [a, b] \to \mathbb{C}$ is square integrable on $[a, b]$ if and only if

$$\int_a^b \left|f(x)\right|^2 dx < \infty \,.$$

The function featuring in Gabriel's Horn, $f(x) = \frac{1}{x}$, is square-integrable on $[1, \infty)$ since $\int_1^{\infty} \frac{1}{x^2}\, dx = 1 < \infty$ the absolute value of its square is finite on $[1, \infty)$. The volume of rotation is π times the integral $\int_a^b \left|f(x)\right|^2 dx$, so the volume of rotation is finite exactly when the function involved is square-integrable.

Being square-integrable does however not guarantee that a function has finite area. Even though $f(x) = \frac{1}{x}$ is square integrable, its area is

$$\int_1^{\infty} \frac{1}{x}\, dx = \infty \,.$$

Thus, as Gabriel's Horn illustrates, a function can have infinite area despite having a finite rotated volume. The opposite can be true of functions that are not square-integrable. For instance, the function $f(x) = \frac{1}{\sqrt{x}}$ on the interval $[0, \pi]$ has finite area but is not square-integrable and therefore has infinitely large rotated volume (this is a great problem to try yourself!).

This article gave a nice example of a function with both finite area and finite rotated volume. I hope it leaves you wondering what other similar functions are out there.

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