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Solutions 1731–1740

Q1731 Let $n = 1204$. The factors of n which lie between $\sqrt{n}$ and n are

$$43\,,\ 86\,,\ 172\,,\ 301\,,\ 602\,,$$

and if we add these up we get our original number, $43 + 86 + 172 + 301 + 602 = 1204$. The same thing works for $n = 1316$. Find (without asking a computer to do it for you!) a number between 1204 and 1316 which has the same property.

SOLUTION By carefully studying the given example, we realise that the reason it works is that $1204 = 28 \times 43$, and 28 is a *perfect number* and 43 is prime. (And $43 > 28$: you may check that it *does not* work if, for example, $n = 28 \times 23$.) Likewise, $1316 = 28 \times 47$. So we want a number between 1204 and 1316 which is a perfect number times a prime. The perfect number cannot be 28, as there are no primes between 43 and 47; and higher perfect numbers are far too large. So we go back to the previous perfect number, 6: we want a prime p such that

$$1204 < 6p < 1316\,.$$

This gives $201 \le p \le 219$, and the only prime in this range is $p = 211$. So the required number is $6 \times 211 = 1266$.

Comment. It is not true that *every* number with the stated property is a perfect number times a prime: you may care to investigate this further.

Q1732 Suppose that the numbers $a_1, a_2, \ldots, a_n$ are equal to $1, 2, \ldots, n$, but not necessarily in that order. Find the maximum possible value of

$$S = \sum_{k=1}^{n} (k - a_k)^2\,,$$

and the values of the a_k which give this maximum.

SOLUTION We shall use the fact that

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}\,,$$

which is a standard exercise in proof by mathematical induction. Expanding all the squares,

$$S = \sum_{k=1}^{n} k^2 - 2\sum_{k=1}^{n} k a_k + \sum_{k=1}^{n} a_k^2\,.$$

Now, the numbers a_k are just $1, 2, \ldots, n$, possibly in a different order, so

$$\sum_{k=1}^{n} a_k^2 = \sum_{k=1}^{n} k^2$$

and we have

$$S = 2\sum_{k=1}^{n} k^2 - 2\sum_{k=1}^{n} ka_k \ .$$

We need to find the arrangement of $1, 2, \ldots, n$ which gives the minimum value of the second sum. This will occur when the a_k have the values from 1 to n in *decreasing* order. To prove this, note that if the a_k are not in decreasing order, then we must have $a_k < a_{k+1}$ for some k. Compare the sum S_1 we have in this case with the sum S_2 obtained by interchanging a_k and a_{k+1}. We have

$$\begin{aligned} S_1 &= a_1 + 2a_2 + \cdots + ka_k + (k+1)a_{k+1} + \cdots + na_n \\ S_2 &= a_1 + 2a_2 + \cdots + ka_{k+1} + (k+1)a_k + \cdots + na_n \ ; \end{aligned}$$

since most of the terms in the two sums are the same,

$$\begin{aligned} S_1 - S_2 &= (ka_k + (k+1)a_{k+1}) - (ka_{k+1} + (k+1)a_k) \\ &= a_{k+1} - a_k \\ &> 0 \ . \end{aligned}$$

That is, $S_2 < S_1$, and so S_1 does not give the minimum value of the sum $a_1 + 2a_2 + \cdots + na_n$. This will apply to any arrangement in which we ever have $a_k < a_{k+1}$, and so the arrangement we require is $a_1 = n$, $a_2 = n-1, \ldots, a_n = 1$; that is, $a_k = n + 1 - k$. The maximum value of S is

$$\begin{aligned} S &= 2\sum_{k=1}^{n} k^2 - 2\sum_{k=1}^{n} k(n+1-k) \\ &= 4\sum_{k=1}^{n} k^2 - 2(n+1)\sum_{k=1}^{n} k \\ &= \frac{4n(n+1)(2n+1)}{6} - 2(n+1)\frac{n(n+1)}{2} \\ &= \frac{(n-1)n(n+1)}{3} \ . \end{aligned}$$

Solution received from Henry Ricardo, New York, who points out that the idea we have used here is an example of a *rearrangement inequality*. Let $x_1, x_2, \ldots, x_n$ be positive real numbers in increasing order, and let $y_1, y_2, \ldots, y_n$ be positive numbers. If we consider all sums

$$x_1z_1 + x_2z_2 + \cdots + x_nz_n$$

in which the numbers $z_1, z_2, \ldots, z_n$ are a rearrangement of $y_1, y_2, \ldots, y_n$, then the maximum value of the sum is obtained when the zs are arranged in increasing order, and the minimum is obtained when the zs are arranged in decreasing order. To prove this, essentially follow the argument in the above solution, or see this *Parabola* article.

Rasul Gasimli also sent a solution including a careful proof that $a_1 > a_2 > \cdots > a_n$.

Q1733 Alain is participating in a motor trial over a fixed distance, where each competitor is allocated a target time and has to drive at a fixed speed in order to reach the finish line exactly on time. Alain has his speed all worked out; but just as he is about to start, he is informed that his time allocation has been decreased by 10% because of financial irregularities by his support team. "No problem", says Alain, "I'll just increase my speed by 10%". And so he did. And at the end of his allocated time, he was still some distance short of the finish. What went wrong?

SOLUTION Let Alain's original time be t, and his original speed v (in suitable units). Then the distance to be travelled is

$$x = vt \ .$$

Since his time was decreased by 10% and his speed increased by 10%, the distance he actually travelled was

$$\bigl(v + 10\%v\bigr)\bigl(t - 10\%t\bigr) = (1.1v)(0.9t) = 0.99vt = 0.99x \ ,$$

which is obviously less than x.

What Alain didn't realise is that in this context, a percentage is always a percentage *of some existing figure*. So a decrease of 10% of some quantity is not compensated for by an increase of 10% of the decreased quantity.

Q1734 How many functions f from $\{\,1,2,3,4,5\,\}$ to $\{\,1,2,\ldots,9,10\,\}$ satisfy the conditions

$$f(1) < f(2) \le f(3) < f(4) \le f(5) \ ?$$

SOLUTION Let

$$x_1 = f(1)\,,\ x_2 = f(2) - f(1)\,,\ x_3 = f(3) - f(2)\,,$$
$$x_4 = f(4) - f(3)\,,\ x_5 = f(5) - f(4)\,,\ x_6 = 10 - f(5)\,.$$

Then $x_1, \ldots, x_6$ are integers satisfying

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 10$$
$$x_1 \ge 1\,,\ x_2 \ge 1\,,\ x_3 \ge 0\,,\ x_4 \ge 1\,,\ x_5 \ge 0\,,\ x_6 \ge 0\,.$$

Conversely, any $x_1, \ldots, x_6$ satisfying these conditions will give a choice of $f(1), \ldots, f(5)$. We can count the number of possibilities for $x_1, \ldots, x_6$ by using the "dots and lines" method. Since $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 10$, imagine a row of 10 dots, combined with 5 lines separating the dots into 6 sections. The number of dots in the first section will be the value of x_1, and so on. For example, the arrangement

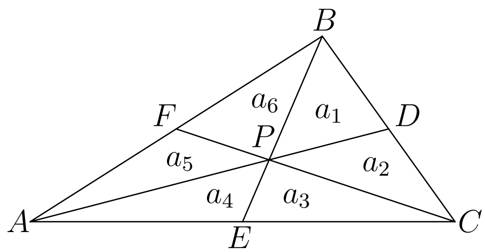
$$\bullet\bullet\,|\,\bullet\,|\,|\,\bullet\bullet\bullet\bullet\bullet\,|\,\bullet\bullet\,|$$

would correspond to the solution $x_1 = 2$, $x_2 = 1$, $x_3 = 0$, $x_4 = 5$, $x_5 = 2$, $x_6 = 0$. Since we require $x_1, x_2, x_4 \ge 1$ we shall reserve one dot for each of sections $1, 2, 4$; it remains to arrange 7 dots and 5 lines in a row. The number of ways of doing so is $C(12, 5) = 792$.

Q1735 Let P be a point inside $\triangle ABC$; let AP, BP, CP meet the sides BC, CA, AB at the points D, E, F, respectively. Show that

$$\frac{|AE|}{|EC|} + \frac{|AF|}{|FB|} = \frac{|AP|}{|PD|} \, .$$

SOLUTION In the diagram, the areas of smaller triangles are denoted by $a_1, \ldots, a_6$ as shown.



The areas of triangles with the same altitude are proportional to their bases. Using this fact in $\triangle APE$, $\triangle EPC$ and also in $\triangle ABE$, $\triangle EBC$, we have

$$\frac{|AE|}{|EC|} = \frac{a_4}{a_3} = \frac{a_4 + a_5 + a_6}{a_1 + a_2 + a_3} \, .$$

Similar arguments give

$$\frac{|AF|}{|FB|} = \frac{a_5}{a_6} = \frac{a_3 + a_4 + a_5}{a_6 + a_1 + a_2}$$

and

$$\frac{|AP|}{|PD|} = \frac{a_5 + a_6}{a_1} = \frac{a_3 + a_4}{a_2} \, . \qquad (*)$$

Now for any positive quantities w, x, y, z, we have the equivalences

$$\frac{w}{x} = \frac{y}{z} \quad \Leftrightarrow \quad wz = xy \quad \Leftrightarrow \quad wz + yz = xy + zy \quad \Leftrightarrow \quad \frac{w+y}{x+z} = \frac{y}{z} \, .$$

Applying this to $(*)$ gives

$$\frac{|AP|}{|PD|} = \frac{a_5 + a_6 + a_3 + a_4}{a_1 + a_2} \, ;$$

and to the previous equations,

$$\frac{|AE|}{|EC|} + \frac{|AF|}{|FB|} = \frac{a_5 + a_6}{a_1 + a_2} + \frac{a_3 + a_4}{a_1 + a_2} = \frac{|AP|}{|PD|}$$

as required.

Q1736 If a polynomial $f(x)$ is divided by $x - a$, the remainder is a constant r; if $f(x)$ is divided by $x - b$, where $b \neq a$, the remainder is s. If $f(x)$ is divided by $(x - a)(x - b)$, then the remainder will be a linear polynomial. Find it.

SOLUTION Write

$$f(x) = (x-a)(x-b)q(x) + (cx+d) \; ; \qquad (*)$$

we seek to find the linear polynomial $cx+d$. Now we have

$$f(x) = (x-a)g(x) + r \; , \quad f(x) = (x-b)h(x) + s$$

for some polynomials $g(x), h(x)$. By equating the first of these expressions with $(*)$ and doing a little algebra, we obtain

$$(x-a)g(x) - (x-a)(x-b)q(x) = (cx+d) - r = c(x-a) + (ca+d-r) \; ,$$

so the polynomial $x-a$ is a factor of the constant $ca+d-r$. The only way this can happen is if the constant is zero: so $ca+d=r$. By a similar procedure, we find $cb+d=s$, and solving these two equations gives

$$c = \frac{r-s}{a-b} \; , \quad d = \frac{as-br}{a-b} \; .$$

Therefore, the remainder polynomial we seek is

$$\frac{r-s}{a-b}\,x + \frac{as-br}{a-b} \; .$$

Solution received from Ibrahim Aghazada, ADA University, Azerbaijan.

Q1737 Find all integers n for which $\sqrt{2024n+1}$ is a positive integer.

SOLUTION Note the factorisation $2024 = 2^3 \times 11 \times 23$. Suppose that $\sqrt{2024n+1} = m$ is a positive integer. This can be written as

$$2024n = m^2 - 1 = (m-1)(m+1) \; ,$$

and it is clear that m must be an odd number. Furthermore, this means that $m-1$ and $m+1$ are two consecutive even numbers, one of them must be a multiple of 4, and so $(m-1)(m+1)$ is divisible by 8. To complete the problem, we need to find all m such that 11×23 is a factor of $(m-1)(m+1)$. This will be so if and only if either one of the factors $m-1$ and $m+1$ is a multiple of 11 and the other is a multiple of 23, or one of them is a multiple of $11 \times 23 = 253$. So there are four cases to consider. We shall use the notation $a \mid b$ to denote that a is a factor of b.

- If $253 \mid m-1$, then we can write $m = 1 + 253s$, where s is an integer. Since we know that m is odd, s must be even, $s = 2t$, and we have $m = 1 + 506t$ for some integer $t \geq 0$.
- Similarly, if $253 \mid m+1$, then we find

  $$m = -1 + 253s = 505 + 253(s-2) = 505 + 506t \; ,$$

  where t is an integer and $t \geq 0$.

- Now suppose that $11 \mid m - 1$ and $23 \mid m + 1$. There is a standard procedure for solving this kind of problem – look up "Chinese Remainder Theorem" – but we shall take a more "low–tech" approach. We write the second statement as $m + 1 = 23r$ and substitute into the first, giving

$$\begin{aligned} 11 \mid 23r - 2 \quad &\Leftrightarrow \quad 11 \mid 22r + (r - 2) \\ &\Leftrightarrow \quad 11 \mid r - 2 \\ &\Leftrightarrow \quad r = 2 + 11s\,, \quad s \in \mathbb{Z}\,. \end{aligned}$$

  Substituting back gives $m = 45 + 253s$; as above, m is odd and so this can be written $m = 45 + 506t$ with $t \geq 0$.
- The remaining case is $11 \mid m + 1$, $23 \mid m - 1$. We invite readers to solve this by using the method of the previous case to show that $m = 461 + 506t$ with $t \geq 0$.

Combining our four solutions gives all possible values for n as

$$n = \frac{(a + 506t)^2 - 1}{2024}\,,$$

where $a = 1$, 45, 461 or 505 and t is an integer, $t \geq 0$.

Solution received: Ilkin Hasanov, ADA University, Azerbaijan, sent an excellent solution using the Chinese Remainder Theorem.

Q1738 Find the smallest set of numbers S which has the properties

- 1 is in S;
- if a, b are any numbers in S, then $1/(a + b)$ is also in S.

SOLUTION The smallest possible set S is the set of all fractions (rational numbers) from $\frac{1}{2}$ to 1, inclusive. To prove this, we have to show that S has the stated properties; and also, that any set satisfying these properties must include every element of S.

The first part is very easy: it is clear that 1 is in S; and if we take any two fractions from $\frac{1}{2}$ to 1, then their sum is a fraction from 1 to 2 and the reciprocal of the sum is a fraction from $\frac{1}{2}$ to 1.

Conversely, let T be any set having the stated properties; we need to show that T includes every fraction from $\frac{1}{2}$ to 1. We shall do this by using induction on q to prove the statement

"T contains every fraction with denominator q between $\frac{1}{2}$ and 1".

This is certainly true when $q = 1$, for the only relevant fraction is $\frac{1}{1} = 1$, and this is in T by assumption.

Now consider a fraction p/q from $\frac{1}{2}$ to 1 with $q \geq 2$, and suppose we already know that T contains all fractions from $\frac{1}{2}$ to 1 having denominator smaller than q. Since we also already know that 1 is in T, we may assume that $p/q < 1$ and so $p < q$.

We study separately the cases when q is odd and when q is even.

If q is even, say $q = 2r$, then we have

$$\frac{1}{2} \leq \frac{p}{2r} < 1 \ ,$$

so $r \leq p < 2r$, which can be written as

$$\frac{1}{2} < \frac{r}{p} \leq 1 \ .$$

Since $p < q$, we know that r/p is in T, and therefore so is

$$\frac{1}{(r/p) + (r/p)} = \frac{p}{2r} = \frac{p}{q} \ .$$

So this case is finished. If q is odd, then let $q = 2r + 1$. Again, we have

$$\frac{1}{2} \leq \frac{p}{2r+1} < 1 \ ,$$

so $2r + 1 \leq 2p$ and $p < 2r + 1$. Now since $2r + 1$ is odd and $2p$ is even, it follows from the first of these that $2r + 2 \leq 2p$; since p and $2r + 1$ are both integers, it follows from the second that $p \leq 2r$. Therefore, we have

$$\frac{1}{2} \leq \frac{r}{p} < \frac{r+1}{p} \leq 1 \ .$$

Once again we recall that $p < q$: so we know that T contains both r/p and $(r + 1)/p$, and hence also contains

$$\frac{1}{(r/p) + \big((r+1)/p\big)} = \frac{p}{2r+1} = \frac{p}{q} \ .$$

By mathematical induction, T contains every fraction from $\frac{1}{2}$ to 1. Thus, S is the smallest possible set having the given properties.

Q1739 A sequence is defined by $a_1 = 1$, $a_2 = m$ and

$$a_{n+1} = \frac{a_n^2 - 1}{a_{n-1}}$$

for $n \geq 2$. Here, m is a fixed integer. Prove that every term a_n is an integer.

SOLUTION It is not hard to check that the first four terms of the sequence are integers: this is given for a_1, a_2, and then we have

$$a_3 = m^2 - 1 \ , \quad a_4 = \frac{(m^2-1)^2 - 1}{m} = m^3 - 2m \ .$$

We shall prove that if the values of four successive terms in the sequence are known to be integers, then the next term is also an integer: it will follow by induction that every term in the sequence is an integer.

So, suppose that n is an integer, $n \geq 4$ and that $a_{n-3}, a_{n-2}, a_{n-1}$ and a_n are integers. From the given recurrence, we have

$$a_{n-2}a_n = a_{n-1}^2 - 1\ , \quad a_{n-3}a_{n-1} = a_{n-2}^2 - 1\ ,$$

and therefore

$$a_{n-2}^2(a_n^2 - 1) = a_{n-1}^4 - 2a_{n-1}^2 - a_{n-3}a_{n-1} = a_{n-1}(a_{n-1}^3 - 2a_{n-1} - a_{n-3})\ .$$

Therefore, a_{n-1} is a factor of $a_{n-2}^2(a_n^2 - 1)$. But the equation

$$a_{n-1}^2 - a_n a_{n-2} = 1$$

implies that a_{n-1} and a_{n-2} have no common factor (any common factor would also be a factor of 1); therefore, a_{n-1} is a factor of $a_n^2 - 1$, and so

$$a_{n+1} = \frac{a_n^2 - 1}{a_{n-1}}$$

is an integer.

Alternative solution. We prove by induction that, if $n \geq 2$, then

$$a_{n+1} = ma_n - a_{n-1}\ . \qquad (*)$$

First, we can use expressions calculated in our previous solution to see that for $n = 2$ and $n = 3$ this statement says

$$m^2 - 1 = m(m) - 1 \quad \text{and} \quad m^3 - 2m = m(m^2 - 1) - m\ ,$$

both of which are clearly true. Suppose that $(*)$ is true for two consecutive integers $n - 1$ and n; and note that from the given recurrence,

$$a_{n+2} = \frac{a_{n+1}^2 - 1}{a_n} \quad \text{and} \quad a_n = \frac{a_{n-1}^2 - 1}{a_{n-2}}\ .$$

Then we have

$$\begin{aligned} a_{n+2} &= \frac{a_{n+1}^2 - 1}{a_n} \\ &= \frac{(ma_n - a_{n-1})^2 - 1}{a_n} \\ &= m^2 a_n - 2ma_{n-1} + \frac{a_{n-1}^2 - 1}{a_n} \\ &= m^2 a_n - 2ma_{n-1} + a_{n-2} \\ &= m(ma_n - a_{n-1}) - (ma_{n-1} - a_{n-2}) \\ &= ma_{n+1} - a_n\ , \end{aligned}$$

so that $(*)$ is also true for $n + 1$. It follows by induction that $(*)$ is true for all n, and it is then clear that every term of the sequence is an integer.

Q1740 Let a be an integer. Find the number of integers b such that the quadratic

$$(x+a)(x+b)+2024$$

can be factorised as the product of two linear factors with integer coefficients.

SOLUTION Since a is an integer, we can write the factorisation as

$$(x+a)(x+b)+2024=(x+a+r)(x+a+s)\ . \qquad (*)$$

Expanding and equating coefficients gives

$$a+b=2a+r+s\ ,\quad ab+2024=(a+r)(a+s)\ .$$

Solving the first equation for b, then taking the second equation minus a times the first and simplifying, yields

$$b=a+r+s\ ,\quad rs=2024\ .$$

Conversely, if these conditions hold, then it is routine to check that we have the factorisation $(*)$. Therefore, the number of possible values for r is the number of factors of 2024; and each r gives one possibility for s and hence one for b. Since the prime factorisation of 2024 is $2^3\times 11^1\times 23^1$, the number of positive factors of 2024 is $(3+1)(1+1)(1+1)=16$, and the total number of factors (for r and s could be negative) is twice this. So there are 32 possibilities for (r,s). However, interchanging r and s gives a different possibility for (r,s), but the same possibility for $b=a+r+s$. So the number of possibilities for b is half of 32, that is, 16.

If you need some explanation of the formula we used for counting the divisors of 2024, then search online for "number of divisors formula".