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PROBLEM SECTION

Q660-Q662 are similar to questions which were used in a mathematics competition for first year university students in Queensland a decade ago. Q663 has appeared in the Canadian magazine Crux Mathematicorium and Q664-Q666 appear in the Russian magazine Kvant. Q667 was used in a 1981 Hungarian competition.

<u>Q. 660</u> A jailer walks n times past a row of n cells walking always from left to right. Under the terms of a partial amnesty, the following happens.

Initially all cells are locked.

On the first pass, he turns every lock, beginning with the first.

On the second pass, he turns every second lock, beginning with the second.

And so on.

More succinctly:-

On the kth pass he turns every kth lock beginning with the kth ($k = 1, 2, \ldots, n$).

Which cells are now unlocked? Prove your assertion.

<u>Q. 661</u> Let $P_n(x)$ be the polynomial

$$P_n(x) = 1 + 2x + 3x^2 + \ldots + (n + 1)x^n.$$

Show that $P_n(x)$ has no real zero if n is even, and exactly one real zero if n is odd, lying between -1 and 0.

<u>Q. 662</u> Let N be the number of strings of 20 letters all either 0 or X, which contain the two letter work 0X precisely 7 times. (e.g. such a string as XX<u>0X</u>0<u>0X</u><u>0X</u>XX<u>0X</u><u>0X</u>X<u>0X</u><u>0X</u> where the occurrences of 0X have been underlined). Find arguments to show that

a) $N = \binom{7}{7}\binom{13}{7} + \binom{8}{7}\binom{12}{7} + \binom{9}{7}\binom{11}{7} + \binom{10}{7}\binom{10}{7} + \binom{11}{7}\binom{9}{7} + \binom{12}{7}\binom{8}{7} + \binom{13}{7}\binom{7}{7}$

and

b) $N = \binom{21}{15}$

(Note: $\binom{n}{n}$ denotes the binormal coefficient for which ${}^nC_n$ is an alternative notation)

Q. 663 ABCD is a skew quadrilateral (i.e. the vertices do not all line in one plane) such that $\angle ABC = \angle BCD = \angle CDA = 90^\circ$. Prove that $\angle DAB$ is acute.

Q. 664 Each side of a triangle of area 1 unit is divided into three equal parts. The six points of division are the vertices of two triangles whose intersection is a hexagon. Find the area of the hexagon.

Q. 665 The sequence $(a_1, a_2, \ldots, a_n)$ is a permutation (i.e. rearrangement) of $(1, 2, \ldots, n)$

a) Prove that $|a_1 - a_2| + |a_2 - a_3| + \ldots + |a_{n-1} - a_n| + |a_n - a_1| \geq 2n - 2$.

b) For how many distinct permutations of $(1, 2, \ldots, n)$ does equality hold in (a).

Q. 666 Can you find an infinite set of natural numbers, S such that for any subset A of S $\sum_{s \in A} s$ is never a perfect square?

What if "square" is replaced by "cube" in the above, or by kth power, (k being any given integer > 3)?

Q. 667 Let $f(k)$ denote the number of zeros in the decimal representation of the natural number k. Compute $S_n = \sum_{k=1}^{n} 2^{f(k)}$ where $n = 10^{10} - 1$.

Q. 668 Which is larger $8^{8^{8^{\cdot^{\cdot^{\cdot^{8}}}}}}$ (where there are altogether nine 8's in the tower), or $9^{9^{\cdot^{\cdot^{\cdot^{9}}}}}$ (where there are eight 9's)?

Q. 669 The medians to two sides of a triangle meet at right angles. The two sides have lengths a and b units. Find conditions on a and b for this to be possible, and express the length of the third side in terms of a and b.

Q. 670 Show how to construct a quadrilateral ABCD given that the diagonal AC bisects the angle A, and given also the lengths of the four sides.

Q. 671 Four rays in 3-D space are drawn from O to the vertices of a convex quadrilateral. Show that there exists a parallelogram ABCD having one vertex on each ray.

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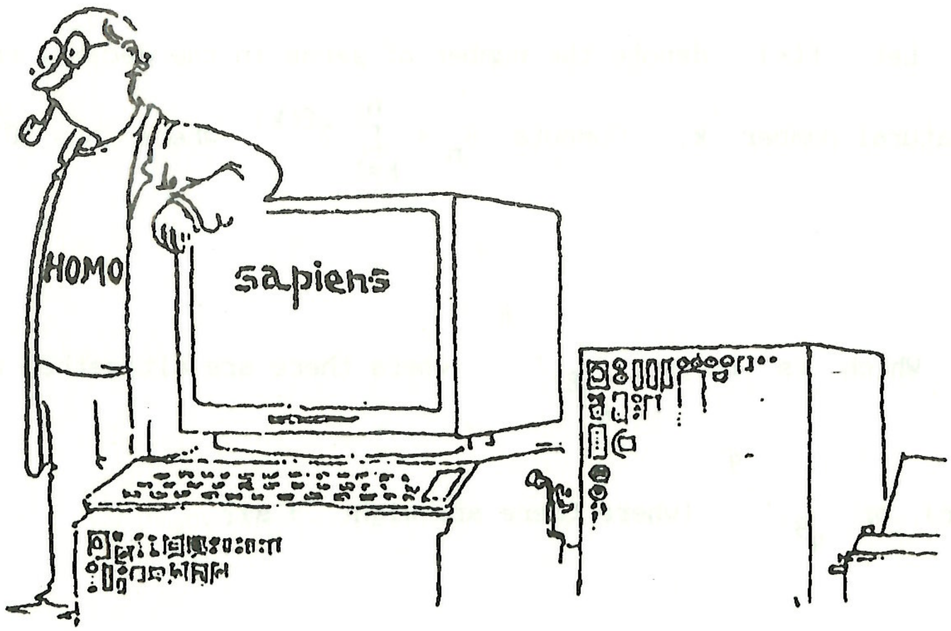

Man the thinker?
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