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A note on the integration of the floor function

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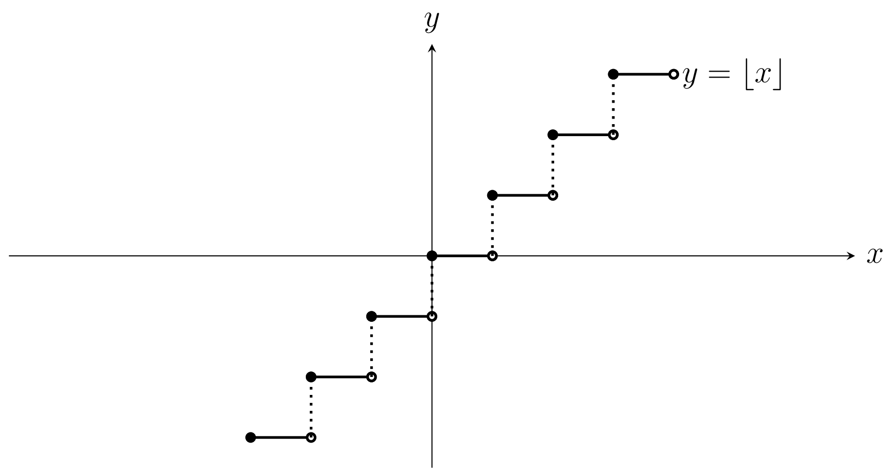
1 Introduction

The *floor function* $\lfloor x \rfloor$ of any real number x is the greatest integer less than or equal to x:



The indefinite integral $\int \lfloor x \rfloor \, dx$ of the floor function does not exist, which is maybe why the definite integral of floor function has not been widely considered. In this note, we evaluate definite integrals of the floor function. Furthermore, we consider definite integrals of $\lfloor x^s \rfloor$ where $s > 0$.

2 The definite integration of the floor function

Let

$$F(x) = \int_0^x \lfloor t \rfloor \, dt \, .$$

We calculate $F(x)$ for x in each of the intervals $(-\infty, -1)$, $[-1, 0)$, $[0, 1)$ and $[1, \infty)$.
For $x \in (-\infty, -1)$,

$$F(x) = -\int_x^0 \lfloor t \rfloor \, dt = -\int_x^{\lfloor x \rfloor + 1} \lfloor t \rfloor \, dt - \sum_{n=\lfloor x \rfloor + 1}^{-1} \int_n^{n+1} \lfloor t \rfloor \, dt$$

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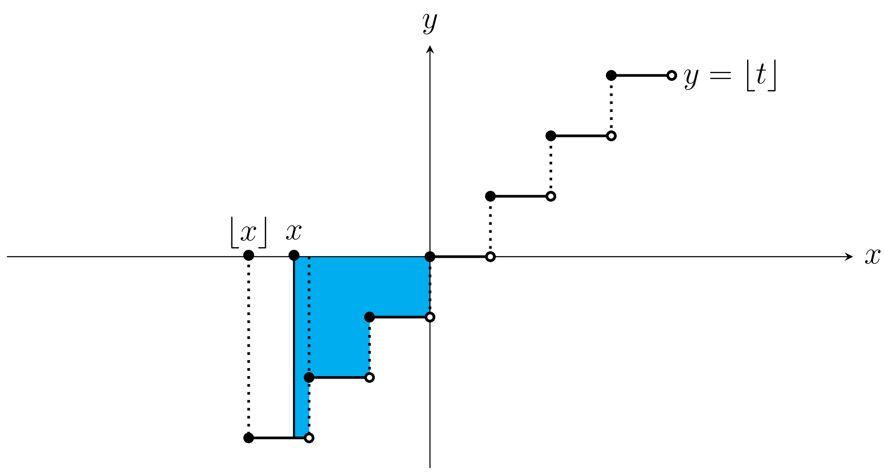
$$
\begin{aligned}
&= -\int_x^{\lfloor x\rfloor+1} \lfloor x\rfloor\, dt - \sum_{n=\lfloor x\rfloor+1}^{-1} \int_n^{n+1} n\, dt \\
&= -\lfloor x\rfloor\big(\lfloor x\rfloor - x + 1\big) + \sum_{n=1}^{-1-\lfloor x\rfloor} n \\
&= -\lfloor x\rfloor\big(\lfloor x\rfloor - x + 1\big) + \frac{1}{2}\big(-1-\lfloor x\rfloor\big)\big(-\lfloor x\rfloor\big) \\
&= x\lfloor x\rfloor - \frac{1}{2}\lfloor x\rfloor^2 - \frac{1}{2}\lfloor x\rfloor\,.
\end{aligned}
$$



For $x \in [-1,0)$, we have $\lfloor x\rfloor = -1$, so

$$
F(x) = -\int_x^0 \lfloor t\rfloor\, dt = -\int_x^0 -1\, dt = -x = x\lfloor x\rfloor - \frac{1}{2}\lfloor x\rfloor^2 - \frac{1}{2}\lfloor x\rfloor\,.
$$

For $x \in [0,1)$, we have $\lfloor x\rfloor = 0$, so

$$
F(x) = \int_0^x \lfloor t\rfloor\, dt = \int_0^x 0\, dt = 0 = x\lfloor x\rfloor - \frac{1}{2}\lfloor x\rfloor^2 - \frac{1}{2}\lfloor x\rfloor\,.
$$

For $x \in (1,\infty)$,

$$
\begin{aligned}
F(x) = \int_0^x \lfloor t\rfloor\, dt = \sum_{n=0}^{\lfloor x\rfloor-1} \int_n^{n+1} \lfloor t\rfloor\, dt + \int_{\lfloor x\rfloor}^x \lfloor t\rfloor\, dt &= \sum_{n=0}^{\lfloor x\rfloor-1} \int_n^{n+1} n\, dt + \int_{\lfloor x\rfloor}^x \lfloor x\rfloor\, dt \\
&= \sum_{n=0}^{\lfloor x\rfloor-1} n + \lfloor x\rfloor\big(x - \lfloor x\rfloor\big) \\
&= \frac{1}{2}\big(\lfloor x\rfloor - 1\big)\lfloor x\rfloor + \lfloor x\rfloor\big(x - \lfloor x\rfloor\big) \\
&= x\lfloor x\rfloor - \frac{1}{2}\lfloor x\rfloor^2 - \frac{1}{2}\lfloor x\rfloor\,.
\end{aligned}
$$

In each case,

$$F(x) = x\lfloor x\rfloor - \frac{1}{2}\lfloor x\rfloor^2 - \frac{1}{2}\lfloor x\rfloor\,.$$

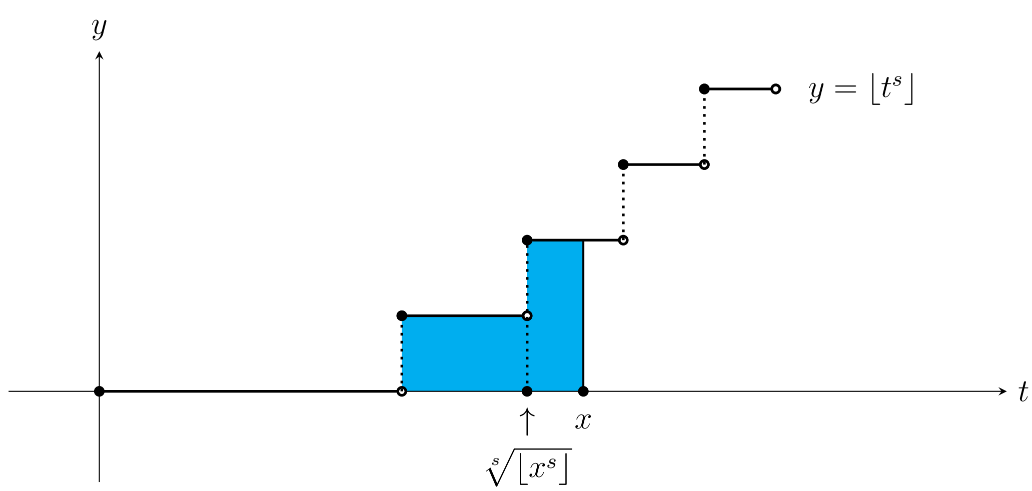
Therefore, the definite integral of the floor function is as follows:

$$\begin{aligned}\int_a^b \lfloor x\rfloor\,dx &= \int_0^b \lfloor x\rfloor\,dx - \int_0^a \lfloor x\rfloor\,dx = F(b) - F(a) \\ &= \left(b\lfloor b\rfloor - \frac{1}{2}\lfloor b\rfloor^2 - \frac{1}{2}\lfloor b\rfloor\right) - \left(a\lfloor a\rfloor - \frac{1}{2}\lfloor a\rfloor^2 - \frac{1}{2}\lfloor a\rfloor\right)\,. \qquad (1)\end{aligned}$$

3 The definite integral of the function $\lfloor x^s\rfloor$

Using the same approach as above, we can calculate the definite integrals of $\lfloor x^s\rfloor$ for $x > 0$ where $s > 0$. To do so, we define and evaluate

$$F_s(x) = \int_0^x \lfloor t^s\rfloor\,dt\,.$$



For $x \in [1, \infty)$,

$$\begin{aligned}F_s(x) = \int_0^x \lfloor t^s\rfloor\,dt &= \sum_{n=0}^{\lfloor x^s\rfloor - 1} \int_{\sqrt[s]{n}}^{\sqrt[s]{n+1}} \lfloor t^s\rfloor\,dt + \int_{\sqrt[s]{\lfloor x^s\rfloor}}^{x} \lfloor t^s\rfloor\,dt \\ &= \sum_{n=0}^{\lfloor x^s\rfloor - 1} \int_{\sqrt[s]{n}}^{\sqrt[s]{n+1}} n\,dt + \int_{\sqrt[s]{\lfloor x^s\rfloor}}^{x} \lfloor x^s\rfloor\,dt \\ &= \sum_{n=0}^{\lfloor x^s\rfloor - 1} n\left(\sqrt[s]{n+1} - \sqrt[s]{n}\right) + \lfloor x^s\rfloor\left(x - \sqrt[s]{\lfloor x^s\rfloor}\right) \\ &= \sum_{n=0}^{\lfloor x^s\rfloor - 1} \left((n+1)\sqrt[s]{n+1} - \sqrt[s]{n+1} - n\sqrt[s]{n}\right) + x\lfloor x^s\rfloor - \lfloor x^s\rfloor\sqrt[s]{\lfloor x^s\rfloor}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=1}^{\lfloor x^s \rfloor} n\sqrt[s]{n} - \sum_{n=1}^{\lfloor x^s \rfloor} \sqrt[s]{n} - \sum_{n=0}^{\lfloor x^s \rfloor - 1} n\sqrt[s]{n} - \lfloor x^s \rfloor \sqrt[s]{\lfloor x^s \rfloor} + x\lfloor x^s \rfloor \\
&= x\lfloor x^s \rfloor - \sum_{n=0}^{\lfloor x^s \rfloor} \sqrt[s]{n}\,.
\end{aligned}
$$

For $x \in [0, 1)$, we have $\lfloor x^s \rfloor = 0$, so

$$
F_s(x) = \int_0^x \lfloor t^s \rfloor = 0 = x\lfloor x^s \rfloor - \sum_{n=0}^{\lfloor x^s \rfloor} \sqrt[s]{n}\,.
$$

In either case, the expression for $F_s(x)$ is as stated in the following proposition.

Proposition 1. *For each $x \geq 0$,*

$$
F_s(x) = x\lfloor x^s \rfloor - \sum_{n=0}^{\lfloor x^s \rfloor} \sqrt[s]{n}\,.
$$

Consider the special case in which $s = 1/n$ for some positive n:

$$
F_{1/n}(x) = x\lfloor x^{1/n} \rfloor - \sum_{k=0}^{\lfloor x^{1/n} \rfloor} k^n
$$

Using Bernoulli polynomials [1] yields the following proposition:

Proposition 2. *For each $n \in \mathbb{N}$,*

$$
F_{1/n}(x) = x\lfloor x^{1/n} \rfloor - \frac{B_{n+1}(\lfloor x^{1/n} \rfloor + 1) - B_{n+1}(0)}{n+1}
$$

where $B_n(x)$ is the nth Bernoulli polynomial.

The second Bernoulli polynomial is $B_2(x) = x^2 - x + \frac{1}{6}$. Therefore, Equation (1) is retrieved by setting $n = 1$ in Proposition 2:

$$
F(x) = F_1(x) = x\lfloor x \rfloor - \frac{B_2\big(\lfloor x \rfloor + 1\big) - B_2(0)}{2} = x\lfloor x \rfloor - \frac{1}{2}\lfloor x \rfloor^2 - \frac{1}{2}\lfloor x \rfloor\,.
$$

The definite integrals of $\lfloor x^s \rfloor$ can now be calculated from $F_s(x)$:

$$
\int_a^b \lfloor x^s \rfloor \, dx = \int_0^b \lfloor x^s \rfloor \, dx - \int_0^a \lfloor x^s \rfloor \, dx = F_s(b) - F_s(a)\,.
$$

Propositions 1 and 2 can be used to calculate this integral more explicitly.

4 Summary

In this note, we evaluated the definite integral of $\lfloor x \rfloor$ using a geometric argument. We also calculated the definite integrals of $\lfloor x^s \rfloor$.

Using geometric arguments such as those in this note, many definite integrals can be evaluated, even if their indefinite integral doesn't exist.

Acknowledgements

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References

[1] E.W. Weisstein, Bernoulli Polynomial, *MathWorld–A Wolfram Web Resource*, https://mathworld.wolfram.com/BernoulliPolynomial.html,
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