

Recursive sequences and solution techniques

Diya Gandhi¹

1 Introduction

This paper provides different techniques for solving maths competition problems that involve sequences. Some of these problems can be solved by the periodicity technique; see Section 2. Others can be solved by finding an explicit formula; see Section 3. However, some sequence problems cannot easily be solved in these ways, and other solution techniques are then more useful; see Section 4.

2 The periodicity technique

Recursive formulas give the value of a specific term based on previous terms, while explicit formulas give the value of a specific term based on its position. In some cases, it will be difficult to find any explicit formula to solve the problem. For such cases, the most common approach to solve the problem is to apply the *periodicity technique*. This technique is simply to use a recursive formula to find the first few terms in the sequence until you find a repeating pattern. Then, you use the pattern to find the next terms of the sequence.

Problem 1. [5] (Problem 2/3/25 USA Mathematical Talent Search 2013–2014)

Let a_1, a_2, a_3, \dots be a sequence of positive real numbers such that $a_k a_{k+2} = a_{k+1} + 1$ for all k . Suppose that a_1 and a_2 are positive integers. Find the maximum possible value of a_{2014} .

Solution. The periodicity technique is used to solve this problem. Write $a_1 = a$ and $a_2 = b$, where a and b are positive integers. By the recursive formula, $a_1 a_3 = a_2 + 1$, so $a_3 = \frac{a_2+1}{a_1} = \frac{b+1}{a}$. Similarly, $a_4 = \frac{a+b+1}{ab}$, $a_5 = \frac{a+1}{b}$, $a_6 = a = a_1$ and $a_7 = b = a_2$. The periodicity of the sequence is then 5, so $a_{2014} = a_4 = \frac{a+b+1}{ab}$. To get the maximum value of $\frac{a+b+1}{ab}$, ab has to be the smallest value possible. Since a and b are positive integers, $ab = 1$. So, the maximum possible value of a_{2014} is 3.

Problem 2. [3]

Let (x_n) be a sequence defined by $x_1 = 1$, $x_2 = x_3 = \frac{1}{2}$ and $x_{n+3} = 2x_{n+2}x_{n+1} - x_n$ for $n \geq 4$. How many three-digit numbers n are there such that $x_n = 1$?

¹Diya Gandhi is a high school student at Buckingham Browne and Nichols.

Solution. Let us calculate some terms of the sequence (x_n) using the recursion formula. For instance, $x_4 = 2x_3x_2 - x_1 = 2 \cdot \frac{1}{2} \cdot \frac{1}{2} - 1 = -\frac{1}{2}$. More terms are calculated similarly:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}	x_{15}
1	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	-1	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	-1	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$

We see that this sequence will repeat after x_{12} with a periodicity of 12. Therefore, $x_n = 1$ when $n - 1$ is divisible by twelve. The first 1 to appear in the sequence when $n \geq 100$ is x_{109} . Starting at $n = 109$ and ending before n becomes a four-digit number, you will get another 1 in the sequence. You can do this 74 times before n is greater than 1000. Including x_{109} , there are 75 values of n such that $x_n = 1$.

Problem 3. [1]

What is the remainder when the 2019th term of the Fibonacci sequence is divided by 7?

Solution. The 2019th term of the Fibonacci sequence is too much work to manually calculate by brute force. Instead, it is better to look for a pattern. By using the recursive identity $F_{n+2} = F_{n+1} + F_n$ for Fibonacci numbers together with modular arithmetic where we divide each number F_{n+2} by 7 and look at the remainder, we can quickly find the sequence of these remainders. Here are the 32 first such remainders:

1 1 2 3 5 1 6 0 6 6 5 4 2 6 1 0 1 1 2 3 5 1 6 0 6 6 5 4 2 6 .

We see that the 16 remainders 1, 1, 2, 3, 5, 1, 6, 0, 6, 6, 5, 4, 2, 6, 1, 0 are repeated in this list, and the recursive identity ensures that they continue to repeat throughout the entire Fibonacci sequence. Next, we find the remainder 3 when 2019 is divided by the number of terms in the pattern, namely 16. The 3rd term in the pattern is 2 which is the answer.

3 The explicit-formula technique

The explicit-formula technique is to find an explicit formula for a given recursion sequence. Many recursive sequence problems allow us to find recursive formulas. However, there is no one way how to find an explicit formula, and solutions will differ based on the information given in each problem. Here are a few examples of creating an explicit formula.

Problem 4. [3]

Let (a_n) be the sequence defined by $a_1 = 1$, $a_2 = \frac{3}{7}$ and, for all $n \geq 3$,

$$a_n = \frac{a_{n-2}a_{n-1}}{2a_{n-2} - a_{n-1}}.$$

Write a_{2019} as $\frac{p}{q}$, where p and q are relatively prime positive integers. What is $p + q$?

Solution. After using the recursive formula to find $a_3 = \frac{3}{11}$ and $a_4 = \frac{3}{15}$, we notice that the formula $a_n = \frac{3}{4n-1}$ is true for $n = 1, 2, 3, 4$. The method of induction is used in order to prove that this explicit formula is true for all $n \geq 1$. Assume that the formula is true for all integers $n = 1, 2, \dots, m$ for some $m \geq 4$. Then $a_{m-1} = \frac{3}{4m-5}$ and $a_m = \frac{3}{4m-1}$, so

$$a_{m+1} = \frac{a_{m-1}a_m}{2\frac{3}{4m-5} - \frac{3}{4m-1}} = \frac{\frac{3}{4m-5} \frac{3}{4m-1}}{2\frac{3}{4m-5} - \frac{3}{4m-1}} = \frac{9}{6(4m-1) - 3(4m-5)} = \frac{3}{4(m+1)} - 1.$$

By induction, the explicit formula $a_n = \frac{3}{4n-1}$ is true for all integers $n \geq 1$. In particular, $a_{2019} = \frac{3}{8075}$. Here, $p = 3$ and $q = 8075$, so $p + q = 8078$.

Problem 5. [2]

Consider the sequence (a_n) defined by $a_1 = 1$, $a_2 = 5$ and, for $n = 1, 2, \dots$,

$$a_{n+2} = 5a_{n+1} - 6a_n.$$

What is the value of $a_{100} - 3a_{99}$?

Solution. We are here given a 2nd-order difference equation and could use the standard formula for such equations. Instead, we show a nice alternative method. In particular, let us define the numbers b_n for $n \geq 1$ as follows:

$$b_n = a_{n+1} - 3a_n.$$

Note that $b_{n+1} = a_{n+2} - 3a_{n+1} = 2(a_{n+1} - 3a_n) = 2b_n$. It follows that, for each $n \geq 1$, $b_n = 2^{n-1}b_1 = 2^{n-1}(a_2 - 3a_1) = 2^{n-1}(5 - 3 \cdot 1) = 2^n$, so

$$a_{100} - 3a_{99} = b_{99} = 2^{99}.$$

4 Other common techniques

Here we will solve recursive-sequences problems using other common techniques than those demonstrated in the previous sections.

Problem 6. [4]

How many terms of the sequence $x_n = 10^n - 3^n + 2^n + 5$ ($n \geq 1$) are perfect squares?

Solution. Note that $3 \mid 2^n + 1$ for odd n and $3 \mid 10^n - 1$ for any n . Thus for any odd n ,

$$x_n = (10^n - 1) - 3^n + (2^n + 1) + 5$$

leaves the remainder 2 when divided by 3. Therefore, x_n is not a perfect square. Now,

$$x_2 = 10^2 - 3^2 + 2^2 + 5 = 10^2,$$

and, for any positive integer k greater than 1,

$$(10^k - 1)^2 < 10^{2k} - 3^{2k} + 2^{2k} + 5 < 10^{2k}.$$

Since x_{2k} is between two consecutive perfect squares, it is not a perfect square. Hence, x_n is a perfect square for exactly one value of n , namely $n = 2$.

Problem 7. [3]

Let (a_n) and (b_n) be increasing arithmetic sequences of integer terms for positive integers n . Suppose that $a_1 = b_1 = 1$ and k is a positive integer greater than 2 such that $\frac{1}{a_k} + \frac{1}{b_k} = \frac{1}{10}$. What is the value of $\frac{1}{a_2} + \frac{1}{b_2}$?

Solution. From the equation $\frac{1}{a_k} + \frac{1}{b_k} = \frac{1}{10}$, we deduce that $(a_k - 10)(b_k - 10) = 100$. It follows that

$$(a_k - 1)(b_k - 1) - 9(a_k - 1) - 9(b_k - 1) = 19.$$

Since (a_n) and (b_n) are arithmetic sequences, $k - 1 \mid a_k - a_1 = a_k - 1$ and $k - 1 \mid b_k - b_1 = b_k - 1$. Therefore, $k - 1 \mid 19$ and so $k > 2$. Hence, $k = 20$, so $a_{20}, b_{20} \geq 20$. By the first equation, $a_{20} = b_{20} = 20$, so $a_2 = b_2 = 2$. It follows that

$$\frac{1}{a_2} + \frac{1}{b_2} = 1.$$

Problem 8. [2]

Consider the sequence (a_k) of positive rational numbers defined by $a_1 = \frac{2020}{2021}$ and, for $k \geq 1$, if $a_k = \frac{m}{n}$ for relatively prime positive integers m and n , then

$$a_{k+1} = \frac{m + 18}{n + 19}.$$

Find the sum of all integers $j \geq 1$ for which a_j can be written as $\frac{t}{t+1}$ for some positive integer t .

Solution. Note that $a_1 = \frac{t}{t+1}$ for $t = 2020$ and that $a_2 = \frac{2038}{2040} = \frac{1019}{1020} = \frac{t}{t+1}$ for $t = 1019$. So, 1 and 2 are both values of j that we are looking for. Now, for $x = 1, 2, \dots$,

$$a_{2+x} = \frac{1019 + 18x}{1020 + 19x}$$

as long as these fractions cannot be simplified, and until that happens, a_{2+x} cannot be written as $\frac{t}{t+1}$ for any integer t . Let us determine the smallest value of x for which $a_{2+x} = \frac{1019+18x}{1020+19x}$ can be simplified. Note that if $1019 + 18x$ and $1020 + 19x$ have a common factor $d > 1$, then that factor also divides $x + 1 = (1020 + 19x) - (1019 + 18x)$. Therefore, $x \neq 1$, since otherwise $d \mid x + 1 = 2$ and so $d = 2$, a contradiction since $1019 + 18 \cdot 2$ is odd. In this way, it is quick and easy to check the possible values of x , and we find that $x = 6$ is the first time that $a_{2+x} = \frac{1019+18x}{1020+19x}$ can be simplified. In particular,

$$a_8 = \frac{1127}{1234} = \frac{161}{162},$$

and $j = 8$ is therefore the next value that we are looking for. Now for $x = 1, 2, \dots$,

$$a_{8+x} = \frac{161 + 18x}{162 + 19x}$$

until $161 + 18x$ and $162 + 19x$ are no longer relatively prime. Again, any common factor of $161 + 18x$ and $162 + 19x$ must also divide the difference $x + 1$. Using this fact to

conduct a quick manual search, we find that $x = 10$ is the smallest number for which $a_{8+x} = \frac{161+18x}{162+19x}$ can be simplified. In particular, $a_{18} = \frac{341}{352} = \frac{31}{32}$, and we have found the value $j = 18$. Next, for $x = 1, 2, \dots$,

$$a_{18+x} = \frac{31 + 18x}{32 + 19x}$$

is not equal to $\frac{t}{t+1}$ for any integer t unless, perhaps, the fraction $\frac{31+18x}{32+19x}$ can be simplified. By similar manual inspection as above, we find that this first happens when $x = 12$. So, $j = 12 + 18 = 30$ is another value found.

Now, $a_{30} = \frac{247}{260} = \frac{19}{20}$, so $a_{x+30} = \frac{19+18x}{20+19x}$ for $x = 1, 2, \dots$ until this fraction can be simplified. However, that will never happen, for the following reason: if $19 + 18x$ and $20 + 19x$ have a common factor $d > 1$, then d also divides $(20 + 19x) - (19 + 18x) = x + 1$. However, this is not possible since $19 + 18x = 1 + 18(x + 1)$ is then not divisible by d , a contradiction. There are therefore no any integers j greater than 30 for which a_j can be reduced, and so no more numbers a_j can be expressed as $\frac{t}{t+1}$ for any integers t . So, the sum of all j values that we have found is $1 + 2 + 8 + 18 + 30 = 59$.

References

- [1] H. Sedrakyan and N. Sedrakyan, *AMC 8 Preparation Book*, independently published, 2021.
- [2] H. Sedrakyan and N. Sedrakyan, *AMC 10 Preparation Book*, independently published, 2021.
- [3] H. Sedrakyan and N. Sedrakyan, *AMC 12 Preparation Book*, independently published, 2021.
- [4] H. Sedrakyan and N. Sedrakyan, *AIME Preparation Book*, independently published, 2022.
- [5] USA Mathematical Talent Search, Problem 2/3/25, *Round 3 Problems Year 25 — Academic Year 2013–2014*, USAMTS, 2014.