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Dots and lines with upper and lower bounds

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1 Introduction

The *dots and lines method*, also known as *the supermarket principle* and as the *stars and bars method*, is a common technique that often appears in various counting problems. At its core, it helps determine the number of non-negative solutions to the equation

$$x_1 + x_2 + \cdots + x_k = n \qquad \text{where} \quad x_1, x_2, \ldots, x_k \geq 0\,.$$

Many counting problems can surprisingly be reduced to this form. This paper will extend the method to cases in which individual x_i values are bounded, before exploring various interesting problems that involve the technique.

2 A motivating example

Here's a classic counting problem.

Problem 1. *A supermarket has an unlimited amount of apples, oranges and pears. How many ways are there to buy ten fruits from supermarket?*

Solution. Consider the 10 fruits purchased as 'dots' lined up in a row. Two 'lines' are then placed between them to divide the dots into three distinct groups representing the apples, oranges and pears, respectively. For instance, the dots and lines in Figure 1 represent the purchase of 2 apples, 3 oranges and 5 pears. Each placement of the lines represents a distinct way of purchasing the ten fruits. Multiple lines are allowed to be placed next to each other: some groups then have zero fruits. Therefore, the solution is equal to the number of ways to arrange 10 dots and 2 lines. That is the number of ways to choose 2 of the 12 symbols to be line-symbols, namely $\binom{12}{2} = 66$.

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Figure 1: This configuration corresponds to buying 2 apples, 3 oranges and 5 pears.

Now that we have a general idea of how to apply the dots and lines method and how it works, we'll obtain a generalised version of the method.

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3 Dots and lines

The dots and lines method relies on the following combinatorial observation:

Theorem 1. *The number of ways to place n indistinguishable dots into k labelled groups is*

$$\binom{n+k-1}{n} = \binom{n+k-1}{k-1}.$$

In Example 1, there are $k = 3$ groups, each labelled by a type of fruit, and $n = 10$ dots, namely the fruits that we buy.

Proof. Given k groups, draw the n dots in a row and place $k-1$ lines between them to indicate the k groups, just as in Example 1. In turn, any such row of n dots and $k-1$ lines presents k groups of dots uniquely. Thus, the number of ways to place n indistinguishable dots into k labelled groups is the same as the number of ways to arrange n dots and $k-1$ lines in a row, which is

$$\frac{(n+k-1)!}{n!(k-1)!} = \binom{n+k-1}{n} = \binom{n+k-1}{k-1}. \qquad \square$$

An useful algebraic expression for Theorem 1 is as follows.

Theorem 2. *The number of non-negative integer sequences $x_1, x_2, \ldots, x_k$ satisfying*

$$x_1 + x_2 + \cdots + x_k = n$$

is

$$\binom{n+k-1}{n} = \binom{n+k-1}{k-1}.$$

Proof. Let each variable x_i represent the number of dots placed in a group of dots labelled "i". Now apply Theorem 1. $\square$

The combinatorial Theorem 1 and the algebraic Theorem 2 are equivalent and can be used interchangeably.

4 Lower bounds

Consider the following problem.

Problem 2. *How many positive integer sequences $x_1, x_2, \ldots, x_k$ satisfy*

$$x_1 + x_2 + \cdots + x_k = n\ ?$$

This problem is different from that solved by Theorem 2 due to the increased lower bounds $x_1, x_2, \ldots, x_k \geq 1$. However, it can be solved with a simple substitution.

Solution. Set $y_i = x_i - 1$ for $i = 1, 2, \ldots, k$. Then the problem becomes that of finding integers $y_1, y_2, \ldots, y_k$ such that $y_i \geq 0$ and

$$(y_1 + 1) + (y_2 + 1) + \cdots + (y_k + 1) = y_1 + y_2 + \cdots + y_k + k = n\,.$$

By making this substitution, the lower bounds on x_i are removed, and the solution is given by Theorem 2:

$$\binom{(n-k)+k-1}{k-1} = \binom{n-1}{k-1}.$$

This solution can be generalised to obtain a formula for any lower bounds of x_i:

Theorem 3. *Let $c_1, c_2, \ldots, c_k$ be integers with sum $C \leq n$. The number of integer sequences $x_1, x_2, \ldots, x_k$ such that $x_i \geq c_i$ for $i = 1, 2, \ldots, k$ and*

$$x_1 + x_2 + \cdots + x_k = n$$

is

$$\binom{n - C + k - 1}{k-1}.$$

Proof. Set $y_i = x_i - c_i$ for $i = 1, 2, \ldots, k$. We want to find the number of integer sequences $y_1, y_2, \ldots, y_k$ such that $y_i \geq 0$ and

$$(y_1 + c_1) + (y_2 + c_2) + \cdots + (y_k + c_k) = y_1 + y_2 + \cdots + y_k + C = n$$

By Theorem 2, the number of solutions is $\binom{(n-C)+k-1}{k-1}$. □

5 Upper bounds

The problems for which the x_i values have upper bounds are much more difficult. This paper will give an Inclusion-Exclusion approach originally found in [1].

Theorem 4. *Let $r_1, r_2, \ldots, r_k$ be integers. The number integer sequences $x_1, x_2, \ldots, x_k$ such that $0 \leq x_i \leq r_i$ for $i = 1, 2, \ldots, k$ and*

$$x_1 + x_2 + \cdots + x_k = n$$

is

$$\sum_{S \subseteq \{1,2,\ldots,k\}} (-1)^{|S|} \binom{n - \sum_{i \in S}(r_i + 1) + (k-1)}{k-1}.$$

Proof. The rationale for using the Inclusion-Exclusion principle is as follows. If we ignore the upper bounds, then the number of solutions is given by Theorem 2, namely $\binom{n+k-1}{k-1}$. To account for the constraints, we first subtract the number of solutions where at least one x_i exceeds r_i. Since this approach double-counts cases where at least two x_i exceed r_i, we then need to add back the number of such cases. This then double-counts cases where at least three x_i exceed r_i; thus we subtract solutions where at least three x_i exceed r_i, and so forth. This process is systematically handled using the Inclusion-Exclusion principle as described below.

Let A_0 denote the set of all solutions to the equation $x_1 + \cdots + x_k = n$, where $x_1, \ldots, x_k$ are non-negative integers. For $j \in \{1, 2, \ldots, k\}$, let A_j be the subset of A_0 where there are j variables x_i that violate the $x_i \leq r_i$ bound; that is, that satisfy $x_i > r_i$. Then, using Inclusion-Exclusion Principle, the number of solutions to Theorem 4 is

$$\sum_{i=0}^{k} (-1)^i |A_i| \,.$$

All that remains is to find $|A_0|, |A_1|, \ldots, |A_k|$. First, $|A_0|$ can be found using Theorem 2:

$$|A_0| = \binom{n+k-1}{k-1}$$

For $|A_1|$, we consider the number of solutions given that $x_i > r_i$ is true for exactly one i. From Theorem 3, we find that to be

$$\binom{n-(r_i+1)+k-1}{k-1}.$$

Note if $n - (r_i + 1) < 0$, then $x_1 + x_2 + \cdots + x_k > n$ since $x_i > r_i$, so there would be no possible solutions. This also applies to the rest of the $|A_i|$ values. Next, we calculate the number of possible choices for x_i, which corresponds to selecting subsets of size 1 from $\{x_1, x_2, \ldots, x_k\}$:

$$|A_1| = \sum_{i=1}^{k} \binom{n-(r_i+1)+(k-1)}{k-1}.$$

For A_2, we can employ a similar logic. For two specific $x_i > r_i$ and $x_j > r_j$, the solution according to Theorem 3 is

$$\binom{n-(r_i+1)-(r_j+1)+k-1}{k-1}.$$

Now, the number of possible choices for x_{i_1} and x_{i_2} corresponds to selecting subsets of size 2 from $\{x_1, x_2, \ldots, x_k\}$. Thus,

$$|A_2| = \sum_{0 \leq i_1 < i_2 \leq k} \binom{n-(r_{i_1}+1)-(r_{i_2}+1)+(k-1)}{k-1}.$$

We continue in this way and get that for a given integer $j \leq k$,

$$|A_j| = \sum_{0 \leq i_1 < i_2 < \cdots < i_j} \binom{n-(r_{i_1}+1)-(r_{i_2}+1)-\cdots-(r_{i_j}+1)+(k-1)}{k-1}.$$

Therefore,

$$\begin{aligned}\sum_{j=0}^{k}(-1)^j|A_j| &= \sum_{j=0}^{k}(-1)^j \sum_{0\le i_1<i_2<\cdots<i_j} \binom{n-(r_{i_1}+1)-\cdots-(r_{i_j}+1)+(k-1)}{k-1} \\ &= \sum_{S\subseteq\{1,2,\ldots,k\}} (-1)^{|S|}\binom{n-\sum_{i\in S}(r_i+1)+(k-1)}{k-1}.\end{aligned}$$

□

Unlike the formula for lower bounds, the formula for upper bounds is very messy. However, when $r_1 = r_2 = \cdots = r_k = r$ the formula simplifies as follows:

Theorem 5. *The number of integer sequences $x_1, x_2, \ldots, x_k$ such that $x_i \le r$ and*

$$x_1 + x_2 + \cdots + x_k = n$$

is

$$\sum_{i=0}^{\lfloor n/(r+1)\rfloor} (-1)^k \binom{k}{i}\binom{n-i(r+1)+(k-1)}{k-1}.$$

This theorem can be applied to the following problem:

Problem 3. *Ian, Chesandu and Kimberley are playing an exciting game of Dungeons & Dragons. They're in quite a difficult campaign against the evil Dragon of the Dungeon and need to deal at least 12 points of damage to achieve victory. Each player rolls a six-sided die, and their total damage is the sum of their individual rolls. What is the probability that they will defeat the dragon?*

Solution. By letting the damage of each die be x_1, x_2, x_3, respectively, the problem statement becomes finding the probability that

$$x_1 + x_2 + x_3 \ge 12$$

where $1 \le x_1, x_2, x_3 \le 6$. First make the substitutions $y_i = x_i - 1$ for $i = 1, 2, 3$; then $(y_1 + 1) + (y_2 + 1) + (y_3 + 1) \ge 12$ or, in other words,

$$y_1 + y_2 + y_3 \ge 9,$$

where $0 \le y_1, y_2, y_3 \le 5$, allowing us to apply Theorem 5. The possible scores range from 12 to 18, so the problem reduces to finding the probability for the following cases:

$$\begin{aligned} y_1 + y_2 + y_3 &= 9 \\ y_1 + y_2 + y_3 &= 10 \\ &\vdots \\ y_1 + y_2 + y_3 &= 15 \end{aligned}$$

For each case, the number of solutions for y_1, y_2, y_3 can be determined using Theorem 5 by letting $n = 9, 10, \ldots, 15$, $k = 3$ and $r = 5$. Dividing this count by the total number of possible outcomes for y_1, y_2, y_3, which is $6^3 = 216$, yields the desired probability. Thus, the total probability is:

$$P(\text{sum is } 9) + \cdots + P(\text{sum is } 15) = \frac{25}{216} + \frac{21}{216} + \frac{15}{216} + \frac{10}{216} + \frac{5}{216} + \frac{3}{216} + \frac{1}{216} = \frac{10}{27}.$$

These results can be verified with a table of values found in [2].

6 Upper and lower bounds combined

Using both Theorem 3 and Theorem 4, it's possible to give a formula where x_i has both an upper and lower bound:

Theorem 6. *Let $r_1, r_2, \ldots, r_k$ and $c_1, c_2, \ldots, c_k$ be integers with $r_i \geq c_i$. The number of integer sequences $x_1, x_2, \ldots, x_k$ such that $r_i \geq x_i \geq c_i$ for $i = 1, 2, \ldots, k$ and*

$$x_1 + x_2 + \cdots + x_k = n$$

is

$$\sum_{S \subseteq \{1,2,\ldots,k\}} (-1)^{|S|} \binom{n - \sum_{i=1}^{k} c_i - \sum_{i \in S}(r_i - c_i + 1) + (k-1)}{k-1}.$$

Proof. By setting $y_i = x_i - c_i$ for $i = 1, 2, \ldots, k$, we would like to find the number of sets of integers $(y_1, y_2, y_3, \ldots, y_k)$ such that $r_i - c_i \geq y_i \geq 0$ and

$$(y_1 + c_1) + (y_2 + c_2) + \cdots + (y_k + c_k) = y_1 + y_2 + \cdots + y_k + \sum_{i=1}^{k} c_i = n.$$

Now apply Theorem 4. □

7 Interesting problems

Problem 4. *How many tuples of positive integers (v, w, x, y, z) are there such that*

$$v + w + x + y + z = 26$$

and four of them are odd?

Solution. Suppose that z is the only even number and let v, w, x, y be the four odd integers. We can express them in terms of non-negative integers as follows:

$$v = 2a + 1, \quad w = 2b + 1, \quad x = 2c + 1, \quad y = 2d + 1 \quad z = 2e + 2$$

where a, b, c, d, e are non-negative integers. Substituting them into the original equation, it becomes

$$(2a+1)+(2b+1)+(2c+1)+(2d+1)+(2e+2)=26$$

which simplifies to become

$$a+b+c+d+e=10\,.$$

Notice that the solution simply becomes the number of non-negative integer solutions to the equation, which is found by Theorem 2:

$$\binom{10+5-1}{5-1}=\binom{14}{4}\,.$$

Finally, since the choice of z as the even integer is arbitrary, the solution has to be multiplied by $\binom{5}{1}$ since there are $\binom{5}{1}$ ways to select the even variable:

$$\binom{14}{4}\binom{5}{1}=5005\,.$$

Problem 5. *How many* 6*-digit numbers non-decreasing digits when read from left to right?*

Solution. To represent a number like 145557, we first consider the digits from 1 to 9. Place 6 dividing lines after each digit in the number:

$$1|234|5|||67|89\,.$$

Note that there will never be a line before the 1, so the number 1 can be ignored:

$$|234|5|||67|89\,.$$

Each different placement of the bars uniquely represents a unique valid number, and the number of placements can be found using Theorem 1 by letting the digits $2-9$ be dots and the 6 digits of the number be lines. Thus, the solution is:

$$\binom{8+6}{6}=3003\,.$$

Problem 6. *How many distinct terms are there in the expansion of* $(a+b+c+d)^{24}$*?*

Solution. Each term in the expansion corresponds to a distinct product of powers of a, b, c and d, namely $a^{x_1}b^{x_2}c^{x_3}d^{x_4}$, where $x_1+x_2+x_3+x_4=24$, and x_1, x_2, x_3, x_4 are non-negative integers. Thus, the problem is equivalent to finding the number of non-negative integer solutions to the equation

$$x_1+x_2+x_3+x_4=24\,.$$

By Theorem 2, the answer is

$$\binom{27}{3}=2925\,.$$

Problem 7 (due to `Bunch-of-cells`, Mathematical Olympiads Discord Server).
How many triangles can be formed using the vertices of a 30-sided regular polygon, such that there are at least three vertices between any two vertices of the triangle?

Solution. Fix a starting vertex A, and let x_1, x_2, x_3 represent the number of vertices between the vertices of the triangle. There are a total of 30 vertices, and since we are selecting three vertices to form the triangle, we can express the equation as $x_1+x_2+x_3 = 30-3 = 27$, given that $x_1, x_2, x_3 \geq 3$. This can be solved using Theorem 3, giving:

$$\binom{27-9+3-1}{3-1} = \binom{20}{2} = 190\,.$$

Since there are $\binom{30}{1} = 30$ possible starting points for A, the total number of triangles would be $30 \times 190 = 5700$. However, we need to divide by 3 to account for the fact that the same triangle can be formed by rotating the vertices, leading to duplicate cases. This, the final answer is

$$5700/3 = 1900\,.$$

Problem 8 (1986 AIME Problems/Problem 13 [3]). *In a sequence of coin tosses, one can keep a record of instances in which a tail is immediately followed by a head, a head is immediately followed by a head, and etc. We denote these by TH, HH, and etc. For example, in the sequence TTTHHTHTTTHHTTH of 15 coin tosses we observe that there are two HH, three HT, four TH, and five TT subsequences. How many different sequences of 15 coin tosses will contain exactly two HH, three HT, four TH, and five TT subsequences?*

Solution. Note that the structure of the final order is T_H_T_H_T_H_T_H_, and there are 4 spots to put the 2 heads in, and 4 spots to put the 5 tails in. By using Theorem 1 separately on the 4 heads and the 5 tails, and multiplying them together, the answer becomes

$$\binom{2+4-1}{2}\binom{5+4-1}{5} = 560\,.$$

Problem 9 (1994 British Mathematical Olympiad Round 2 Question 2 [4]).
How many different triangles with integer side lengths have with perimeter length 1994*?*

Solution. Let a, b and c represent the three side lengths of the triangle. Given that the perimeter is 1994, we have $a+b+c = 1994$, where a, b and c are positive integers. From the triangle inequality, we also know that

$$a+b > c\,, \quad b+c > a\,, \quad a+c > b\,.$$

Starting with the first inequality, if we add c to both sides of $a+b > c$, then we obtain $a+b+c > 2c$, which simplifies to $1994 > 2c$, yielding $996 \geq c$. Applying similar reasoning to the other two inequalities, we find that

$$996 \geq a, b, c \geq 1\,.$$

The problem now reduces to finding the number of integer solutions to $a+b+c = 1994$ where $1 \leq a, b, c \leq 996$. While this can be solved using Theorem 6, a simpler approach is as follows.

First, multiply the inequalities by -1, which gives $-1 \geq a, b, c \geq -996$. Next, define $a' = 996 - a$, and similarly for b' and c'. These transformation give

$$a' + b' + c' = (996 - a) + (996 - b) + (996 - c) = 3 \times 996 - (a + b + c) = 994\,,$$

where $0 \leq a', b', c' \leq 995$. Notice that the upper bound on a', b', c' is now irrelevant, as it exceeds the total sum. Therefore, using Theorem 2, the number of solutions is

$$\binom{994 + 3 - 1}{3 - 1} = 495510\,.$$

To avoid duplicates, we need to subtract the over-counted cases for isosceles triangles. If $a = b$, then the equation becomes $2a + c = 1994$, with $a, c \leq 996$. Since c must be even, the valid pairs (c, a) are $(2, 996), (4, 994), \ldots, (996, 499)$, yielding a total of 498 triangles. There are $\binom{3}{2} = 3$ ways to choose two equal variables, but we subtract the extra two cases to avoid double-counting. Therefore, the final answer is

$$495510 - 2 \times 498 = 494514\,.$$

Problem 10 (due to `noctnight`, Conquer HSC Discord Server)**.**
A point P inside a triangle $\triangle ABC$ is graceful *if exactly 27 rays can be drawn from it, dividing $\triangle ABC$ into 27 smaller triangles of equal areas.*
Determine the total number of graceful points inside a given triangle $\triangle ABC$.

Solution. Consider a point P inside triangle $\triangle ABC$. To divide the triangle into 27 smaller triangles of equal area using rays from P, we note that P must connect to each vertex A, B and C. Drawing segments from P to the vertices creates three initial sections: triangles $\triangle PBC$, $\triangle PCA$ and $\triangle PAB$.

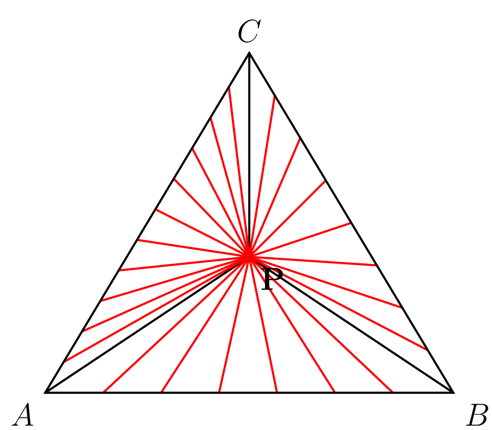

Figure 2: Point P in $\triangle ABC$ and the rays dividing the sections into smaller triangles.

To achieve a total of 27 smaller triangles, the remaining 24 rays must be distributed among the three sections. Let a, b and c be the number of smaller triangles in $\triangle PBC$, $\triangle PCA$ and $\triangle PAB$, respectively. The total number of smaller triangles is

$$a + b + c = 27$$

where a, b and c are positive integers since each section is divided into at least one triangle. The rays must split each triangle into smaller triangles of equal area. The smaller triangles in $\triangle PBC$ all share the same height, namely the perpendicular height from P to line BC. Thus, the base lengths along side BC must be equal, so the a rays divide BC into a equal segments.

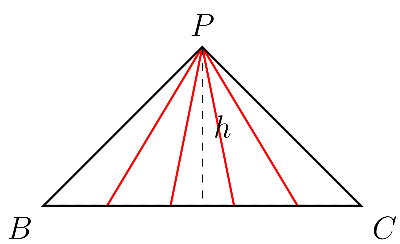

Figure 3: The segment BC is divided into equal parts.

Similarly, the b rays in triangle $\triangle PCA$ divide side CA into b equal segments, and the c rays in triangle $\triangle PAB$ divide side AB into c equal segments. This ensures all smaller triangles have equal areas due to their equal bases and common height.

Finally, the position of point P can be described using barycentric coordinates [2], which are based on the relative areas of triangles $\triangle PBC$, $\triangle PCA$ and $\triangle PAB$ with respect to triangle $\triangle ABC$. Specifically, for any triangle $\triangle ABC$, the barycentric coordinates of a point X are defined as follows:

$$X = \frac{1}{[\triangle ABC]}([\triangle XBC], [\triangle XCA], [\triangle XAB]),$$

where $[\triangle ABC]$ denotes the area of triangle $\triangle ABC$. Using this definition, the barycentric coordinates of P can be expressed as

$$P = \left(\frac{a}{27}, \frac{b}{27}, \frac{c}{27}\right).$$

Notice that the area of each smaller triangle in $\triangle PBC$ is $\frac{1}{27}$ that of $\triangle ABC$, as is each triangle in PCA and PAB, making P a valid graceful point.

Since any point P with barycentric coordinates $\left(\frac{a}{27}, \frac{b}{27}, \frac{c}{27}\right)$ lies inside the triangle, and $\frac{a}{27} + \frac{b}{27} + \frac{c}{27} = 1$ holds, such a point exists for any positive integers a, b and c satisfying $a + b + c = 27$.

[2] More on barycentric coordinates can be found in [5, 6].

Thus the total number of graceful points, is equivalent to finding the number of solutions to $a + b + c = 27$ for positive integers a, b and c. This can be done using Theorem 3, giving the answer

$$\binom{27-3+3-1}{3-1} = 325\,.$$

Thus, the total number of graceful points inside a given triangle $\triangle ABC$ is 325.

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