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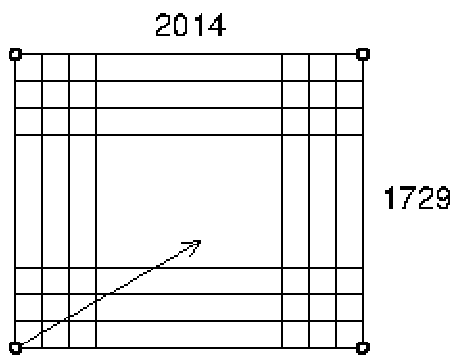
Solutions 1461–1470

Q1461 As in problems 1442 and 1452, a particle is projected from one corner of a 2014×1729 rectangle. This time, however, the particle is projected at an angle of $30°$ above the horizontal. Find which of the four corners, if any, the particle eventually reaches.



SOLUTION At any stage, the distance travelled horizontally is $\sqrt{3}$ times the distance travelled vertically. If the point reaches a corner then the distances travelled horizontally and vertically are both integers; so $\sqrt{3}$ is a ratio of integers. But it is known that this is impossible. Therefore the point, after starting, never reaches any corner.

Q1462 As expected, Mitchell won his dice game with Dale (see problem 1457, solution in the previous issue of *Parabola*). Dale says, "Well of course you had the advantage: you had to get a 1 from a five–sided die and I had to do it from a six–sided die. Not only that, but you got to throw first. Let's play again, with me going first. Then we'll have equal chances." Mitchell replies, "OK, fine, but to make it more interesting we'll both double the number of sides on our dice: I'll use a 10–sided die and you'll use one with 12 sides. You can start, and the first to throw a 1 will win. Obviously, you still have an even chance of winning."

(a) Show that Dale was right in saying that the two had even chances in the game he proposed.

(b) Should he accept Mitchell's alternative offer?

SOLUTION Suppose that the player who throws first uses a die with s sides, and the other uses a die with t sides. If p is the first player's winning probability, then as in the solution to problem 1457 (previous issue) we have

$$p = \frac{1}{s} + \Big(\frac{s-1}{s}\Big)\Big(\frac{t-1}{t}\Big)p\,;$$

this equation is easily solved (exercise!) to give

$$p = \frac{t}{s+t-1} \ .$$

For question (a), we find Dale's winning probability when he goes first by taking $s = 6$ and $t = 5$: this gives $p = \frac{5}{10} = \frac{1}{2}$, so he has an even chance of winning. In case (b) we take $s = 12$ and $t = 10$; Dale's winning probability is $\frac{10}{21}$, which is less than $\frac{1}{2}$, so he should not accept the offer.

Q1463 The positive integers a and b have no common factor. The positive integer n is a multiple of both a and b; exactly half the numbers from 1 to n are multiples of a or of b **or both**. Find a and b.

SOLUTION As in the solution to problem 1456, the number of integers from 1 to n which are multiples of a is n/a, the number which are multiples of b is n/b and the number which are multiples of both is n/ab. In that problem we noted that the latter were counted twice, and were not wanted so should be subtracted twice; in this case we do want to count them, but only once, so we must subtract them once. Thus we have

$$\frac{n}{a} + \frac{n}{b} - \frac{n}{ab} = \frac{n}{2} \ .$$

Algebra similar to that in the previous problem gives

$$(a-2)(b-2) = 2 \ .$$

Since a and b are positive integers, the only possibility is that $a-2$ is 1 and $b-2$ is 2, or vice versa; so a and b are 3 and 4.

Q1464 Suppose that the numbers $x_1, x_2, \ldots, x_{2014}$ form a geometric sequence; and that $y_1, y_2, \ldots, y_{2014}$ is a sequence in which $y_1 = 1$ and $y_{2014} = \frac{1}{2014}$. If

$$x_1^{y_1} = x_2^{y_2} = \cdots = x_{2014}^{y_{2014}} \ ,$$

evaluate the sum

$$\frac{1}{y_1} + \frac{1}{y_2} + \cdots + \frac{1}{y_{2014}} \ .$$

SOLUTION Let the common value of all the numbers $x_j^{y_j}$ be K. Since the x_j form a geometric sequence, any three consecutive values satisfy

$$x_{j-1}x_{j+1} = x_j^2 \ .$$

This can be written as

$$K^{1/y_{j-1}} K^{1/y_{j+1}} = K^{2/y_j} \ ,$$

and therefore

$$\frac{1}{y_{j-1}} + \frac{1}{y_{j+1}} = \frac{2}{y_j} \ .$$

That is, the reciprocals of the numbers y_j form an arithmetic sequence, and their sum is

$$\frac{1}{y_1}+\frac{1}{y_2}+\cdots+\frac{1}{y_{2014}}=\frac{2014}{2}\Big(\frac{1}{y_1}+\frac{1}{y_{2014}}\Big)=2029049\ .$$

Q1465 Show that among any nine people as described in Theorem 3 of the article on "picture proofs" in the previous issue, there must always be four who pairwise like each other or three who pairwise dislike each other.

SOLUTION As in the article, draw a diagram consisting of nine points, with a white line joining two points if the corresponding people like each other, and a black line if they do not. We have to prove that there are either four points for which all the lines joining them are white, or three points for which all the lines joining them are black. Since every point has eight lines attached to it, there are three possibilities:

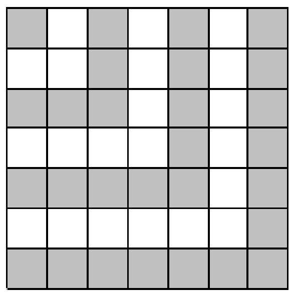
- the point has five white lines and three black lines;
- the point has four or more black lines;
- the point has six or more white lines.

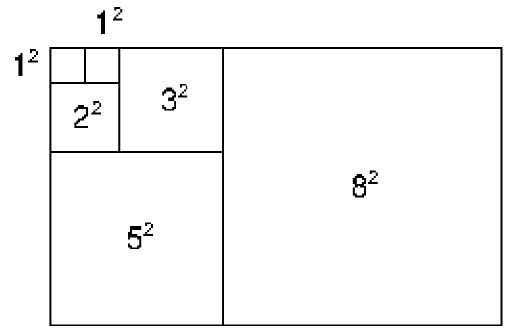
We begin by proving that the first option cannot be true for **all nine** points. If this were the case, counting the two ends of each white line would mean counting 3 at every point, for a total of 27; but as each line has two ends, the total must be even and this is impossible. Therefore, although the first option may well be true in some cases, it cannot be true in *every* case, and we can find a point P at which either the second or third option is true.

If the second option holds, suppose that P is joined to Q, R, S, T by black lines. If any two of Q, R, S, T are joined by a black line, then those two together with P form a set of three points for which all the lines joining them are black; if, on the other hand, no two of Q, R, S, T are joined by black lines, then they are all joined by white lines. In each case we have found the configuration we are seeking.

If the third option holds, say that P is joined to Q, R, S, T, U, V by white lines. If any three of these six points are joined by black lines, we have what we want. If not, then by the theorem proved in the article last issue, three of Q, R, S, T, U, V must be joined by white lines; those three, together with P, form a set of four points for which all the lines joining them are white, and once again we have what we want. This completes the proof.

Q1466 (See the article on "picture proofs" in the previous issue.) What do the following diagrams prove?





SOLUTION In the first diagram, if there are n small squares on each side of the large square, then we have a region of area n^2 split into alternately grey and white regions of area $1, 3, 5, \ldots, 2n-1$. So the diagram shows that

$$1 + 3 + 5 + \cdots + (2n-1) = n^2 \ .$$

Write $F_1, F_2, \ldots$ for the Fibonacci numbers. The second diagram shows that squares of area $1^2, 1^2, 2^2, 3^2, \ldots, F_n^2$ can be fitted together to form a rectangle with sides F_n and $F_n + F_{n-1}$. Remembering also that by definition $F_n + F_{n-1} = F_{n+1}$, this shows that

$$F_1^2 + F_2^2 + F_3^2 + \cdots + F_n^2 = F_n F_{n+1} \ .$$

Q1467 Find two different positive integers m and n such that

$$\lfloor 1000000\sqrt{m} \rfloor = \lfloor 1000000\sqrt{n} \rfloor \ .$$

The notation $\lfloor x \rfloor$ denotes the result of rounding x to the nearest integer downwards, for example, $\lfloor \pi \rfloor = 3$ and $\lfloor 4 \rfloor = 4$.

SOLUTION The simplest way to find a solution will be to make $1000000\sqrt{m}$ an integer and $1000000\sqrt{n}$ just a little more. So we guess $m = k^2$ and $n = k^2 + 1$, and attempt to find a suitable value of k. The left-hand side of the equation is then just $1000000k$, and so we need

$$1000000k < 1000000\sqrt{k^2+1} < 1000000k + 1 \ .$$

The first inequality is clearly true, so we concentrate on the second. It can be rewritten

$$k^2 + 1 < \Big(k + \frac{1}{1000000}\Big)^2 \ ,$$

and expanding and rearranging yields

$$k > \frac{1000000}{2}\Big(1 - \frac{1}{1000000^2}\Big) \ .$$

This suggests that we can take $k = 500000$, which gives $m = 500000^2$ and $n = 500000^2 + 1$; checking, we then have

$$1000000\sqrt{m} = 500000000000 \ , \quad 1000000\sqrt{n} = 500000000000.99 \ ,$$

and rounding down gives the same result for both numbers.

Q1468 Find all solutions of the simultaneous equations

$$x^2 - yz = 1 \ , \quad y^2 - zx = 2 \ , \quad z^2 - xy = -1 \ .$$

SOLUTION Taking x times the third equation, plus y times the first, plus z times the second gives

$$-x + y + 2z = 0 \ . \tag{1}$$

Similarly, taking x times the second, plus y times the third, plus z times the first gives

$$2x - y + z = 0 \ . \tag{2}$$

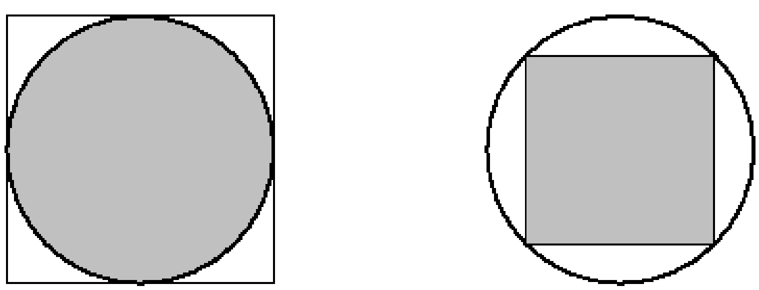
Now add twice equation (1) to equation (2) to get $y + 5z = 0$, so $y = -5z$, and then (1) gives $x = -3z$. Substituting back into the first given equation yields $14z^2 = 1$. So the solutions are

$$x = -\frac{3}{\sqrt{14}} \ , \ y = -\frac{5}{\sqrt{14}} \ , \ z = \frac{1}{\sqrt{14}}$$

and

$$x = \frac{3}{\sqrt{14}} \ , \ y = \frac{5}{\sqrt{14}} \ , \ z = -\frac{1}{\sqrt{14}} \ .$$

Q1469 The diagrams show a circle of radius a with a circumscribed square, and a circle of radius b with an inscribed square.



(a) Show that the ratio of the shaded area to the whole area for the first diagram is greater than $\frac{2}{3}$, and for the second diagram less than $\frac{2}{3}$.

(b) If the shaded area for both parts of the diagram together is exactly $\frac{2}{3}$ of the total area, find the ratio a/b.

(c) If the circles are the same size (that is, $a = b$) and we take m copies of the first diagram and n copies of the second diagram, is it possible to get the shaded area being $\frac{2}{3}$ of the total area?

SOLUTION

(a) Since $\pi > 3$, the two ratios are

$$\frac{\pi a^2}{(2a)^2} = \frac{\pi}{4} > \frac{3}{4} > \frac{2}{3}$$

and

$$\frac{(2b)^2/2}{\pi b^2} = \frac{2}{\pi} < \frac{2}{3}\ .$$

(b) We have

$$\frac{\pi a^2 + 2b^2}{4a^2 + \pi b^2} = \frac{2}{3} \quad \Leftrightarrow \quad (3\pi - 8)a^2 = (2\pi - 6)b^2$$

and so

$$\frac{a}{b} = \sqrt{\frac{2\pi - 6}{3\pi - 8}}\ .$$

(c) To satisfy the given condition we require

$$\frac{m\pi a^2 + 2na^2}{4ma^2 + n\pi a^2} = \frac{2}{3} \quad \Leftrightarrow \quad 3m\pi + 6n = 8m + 2n\pi\ .$$

Treating π as a "variable" and "solving" the equation, we get

$$\pi = \frac{8m - 6n}{3m - 2n}.$$

But this is impossible since π is an irrational number and cannot be expressed as the ratio of two integers.

Q1470 A triangle has sides a, b, c; the angles opposite these sides are A, B, C respectively. Prove that

$$a^3 \sin(B - C) + b^3 \sin(C - A) + c^3 \sin(A - B) = 0\ .$$

SOLUTION By the sine rule we have

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}\ ;$$

denote the common value by K. Using the "sine of a difference" formula we have

$$\begin{aligned} &a^3 \sin(B - C) + b^3 \sin(C - A) + c^3 \sin(A - B) \\ &\quad = a^3(\sin B \cos C - \cos B \sin C) + b^3(\sin C \cos A - \cos C \sin A) \end{aligned}$$

$$
\begin{aligned}
&\quad + c^3(\sin A \cos B - \cos A \sin B) \\
&= a^3(bK \cos C - cK \cos B) + b^3(cK \cos A - aK \cos C) \\
&\quad + c^3(aK \cos B - bK \cos A) \\
&= K\big[(a^2 - b^2)ab \cos C + (b^2 - c^2)bc \cos A + (c^2 - a^2)ca \cos B\big] \ldots
\end{aligned}
$$

...and now we use the cosine rule to show that this equals

$$\frac{K}{2}\big[(a^2 - b^2)(a^2 + b^2 - c^2) + (b^2 - c^2)(b^2 + c^2 - a^2) + (c^2 - a^2)(c^2 + a^2 - b^2)\big] .$$

If we expand this, everything cancels and we get zero, as required.