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The Enclosure

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An enclosure of length 1 unit is constructed around two adjoining walls of unlimited length. It is made of $n \geq 2$ straight sections, referred to as an n-enclosure, designed so as to maximise the enclosed area $A_n(\omega)$, where $\omega \leq \pi$ is the angle formed by the walls.

For $n = 2$, the enclosure's perimeter equals 1 if each wall has length

$$x = \frac{1}{2} \operatorname{cosec}\left(\frac{\omega}{2}\right) \sin\left(\phi + \frac{\omega}{2}\right) .$$

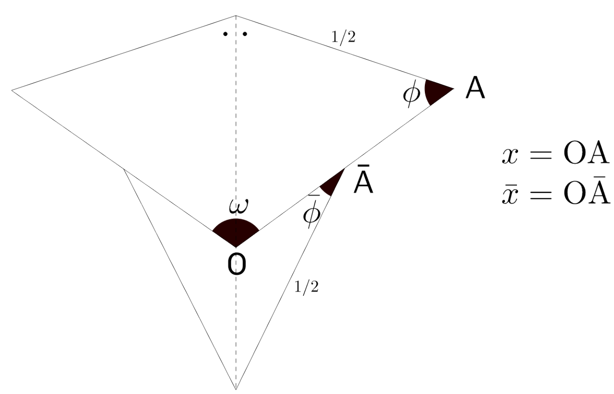

Figure 1: The 2-enclosure.

In terms of the independent variable ϕ,

$$A_2(\omega) = \frac{1}{4} \operatorname{cosec}^2\left(\frac{\omega}{2}\right) \sin\phi \sin\left(\phi + \frac{\omega}{2}\right) ,$$

so

$$\frac{\mathrm{d}A_2(\omega)}{\mathrm{d}\phi} = \frac{1}{4} \operatorname{cosec}^2\left(\frac{\omega}{2}\right) \sin\left(2\phi + \frac{\omega}{2}\right) = 0 .$$

The solution to this equation is $\phi = \frac{\pi}{2} - \frac{\omega}{4}$. The maximum enclosed area is

$$A_2^*(\omega) = A_2(\omega) : \phi = \frac{\pi}{2} - \frac{\omega}{4} = \frac{1}{8} \cot\left(\frac{\omega}{4}\right) .$$

Over the other side of the walls, $\bar{x}$ and $A_2(\bar{\omega})$ where

$$\bar{\omega} := 2\pi - \omega$$

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bear the same expressions as x and $A_2(\omega)$ with ϕ, ω substituted by $\bar{\phi}, \bar{\omega}$. Hence,

$$A_2^*(\bar{\omega}) = \frac{1}{8} \cot\left(\frac{\bar{\omega}}{4}\right) = \frac{1}{8} \tan\left(\frac{\omega}{4}\right) .$$

For $n = 3$, we have $AB = x \sin(\frac{\omega}{2}) = \frac{1}{2} - y(1 - \cos\nu)$.

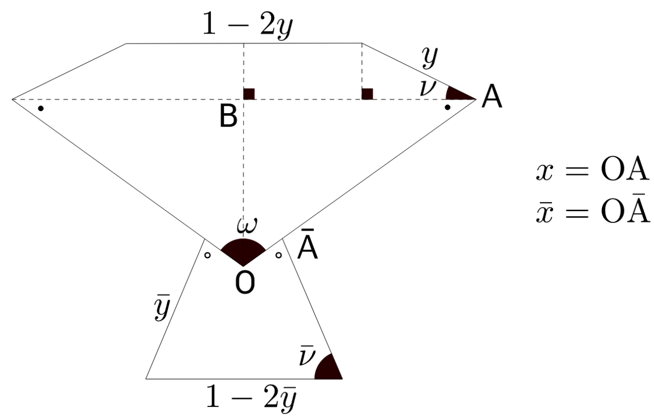

Figure 2: The 3-enclosure.

In terms of the independent variables y, ν

$$A_3(\omega) = \frac{1}{2} y^2 \sin(2\nu) + y(1-2y)\sin\nu + \cot\left(\frac{\omega}{2}\right)\left(\frac{1}{2} - y(1-\cos\nu)\right)^2 ,$$

so

$$\frac{\partial A_3(\omega)}{\partial y} = y\sin(2\nu) + (1-4y)\sin\nu - 2\cot\left(\frac{\omega}{2}\right)(1-\cos\nu)\left(\frac{1}{2} - y(1-\cos\nu)\right) = 0 .$$

Hence,

$$\frac{1}{y}\frac{\partial A_3(\omega)}{\partial \nu} = y\cos(2\nu) + (1-2y)\cos\nu - 2\cot\left(\frac{\omega}{2}\right)\cos\nu\left(\frac{1}{2} - y(1-\cos\nu)\right) = 0 .$$

where ∂ denotes the partial differentiation.

Multiplying the former by $\sin\nu$, the latter by $1-\cos\nu$ and subtracting one expression from the other yields $y = \frac{1}{3}$. Substituting this value into either expression leads to $\nu = \frac{\omega}{3}$. The maximum enclosed area is

$$A_3^*(\omega) = A_3(\omega) : y \mapsto \frac{1}{3}, \nu \mapsto \frac{\omega}{3} = \frac{1}{12}\cot\left(\frac{\omega}{6}\right) .$$

Over the other side of the walls, $\bar{x}, \bar{y}$ and $A_3(\bar{\omega})$ are as per x, y and $A_3(\omega)$ with ν, ω substituted by $\bar{\nu}, \bar{\omega}$. Hence,

$$A_3^*(\bar{\omega}) = \frac{1}{12}\cot\left(\frac{\bar{\omega}}{6}\right) = \frac{1}{12}\cot\left(\frac{\pi}{3} - \frac{\omega}{6}\right) .$$

Recapping this: For either case, the maximum enclosed area is a tessellation by isosceles triangles: 2 triangles of equal sides equal to $\frac{1}{4}\operatorname{cosec}\left(\frac{\omega}{4}\right)$ for $n = 2$, and 3 triangles of side lengths each equal to $\frac{1}{6}\operatorname{cosec}\left(\frac{\omega}{6}\right)$ for $n = 3$.

The common apex of these triangles will subsequently be referred to as the centre of the enclosure. Staying in that vein spawns the following proposition:

"An n-enclosure holds the maximum area if its centre of tessellation, by n isosceles triangles, coincides with the junction of the walls."

The tessellating triangles have base equal to $\frac{1}{n}$ and apex angles $\frac{\omega}{n}$; hence, the maximum area is

$$A_n^*(\omega) = \frac{1}{4n}\cot\left(\frac{\omega}{2n}\right) .$$

By analogy, the area of a regular n-gon of perimeter 1 is

$$H_n = \frac{1}{4n}\cot\left(\frac{\pi}{n}\right) .$$

Thus,

$$A_n^*\left(\frac{2p\pi}{q}\right) = \frac{q}{p}H_{\frac{qn}{p}}, \qquad (p,q) = 1, p \le \frac{q}{2}$$

which is the $\dfrac{p}{q}$-th fraction of a regular $\dfrac{qn}{p}$-gon of perimeter $\dfrac{q}{p}$ for $n = p, 2p, 3p, \ldots$.

For example, $A_5^*(2) \doteqdot A_5^*\left(\frac{5\pi}{8}\right) = \frac{8}{5}H_8$ or $A_7^*(2) \doteqdot A_7^*\left(\frac{7\pi}{11}\right) = \frac{22}{7}H_{22}$.

With $\omega + \bar{\omega} = 2\pi$, we have the following inequalities

- $A_n^*(\bar{\omega}) \le 2H_{2n} \le A_n^*(\omega)$,
- $(A_n^*(\omega)A_n^*(\bar{\omega}))^{\frac{1}{2}} = \frac{1}{4n}\left(\cot\left(\frac{\omega}{2n}\right)\cot\left(\frac{\bar{\omega}}{2n}\right)\right)^{\frac{1}{2}} \ge \frac{1}{4n}\cot\left(\frac{\pi}{2n}\right) = 2H_{2n}$.

These imply that $\frac{A_n^*(\omega)+A_n^*(\bar{\omega})}{2} \ge 2H_{2n}$.

The inequality is reversed with the harmonic means of $A_n^*(\omega), A_n^*(\bar{\omega})$, namely

$$\begin{aligned}\left(\frac{\frac{1}{A_n^*(\omega)} + \frac{1}{A_n^*(\bar{\omega})}}{2}\right)^{-1} &= \frac{1}{2n}\operatorname{cosec}\left(\frac{\pi}{n}\right)\cos\left(\frac{\omega}{2n}\right)\cos\left(\frac{\bar{\omega}}{2n}\right) \\ &\le \frac{1}{2n}\operatorname{cosec}\left(\frac{\pi}{n}\right)\cos^2\left(\frac{\pi}{2n}\right) = \frac{1}{4n}\cot\left(\frac{\pi}{2n}\right) = 2H_{2n},\end{aligned}$$

This is rather interesting!

This article concludes with a shift of the enclosure's centre along the bisector of ω, giving it more "roundness" and visual appeal at the expense of some area.

The enclosure's perimeter is 1 if each wall has length

$$x = \frac{1}{2n}\operatorname{cosec}\left(\frac{\omega}{2}\right)\sin\left(\frac{\theta}{2}\right)\operatorname{cosec}\left(\frac{\theta}{2n}\right)$$

so

$$z = \frac{1}{2n}\operatorname{cosec}\left(\frac{\omega}{2}\right)\sin\left(\frac{\theta-\omega}{2}\right)\operatorname{cosec}\left(\frac{\theta}{2n}\right) ,$$

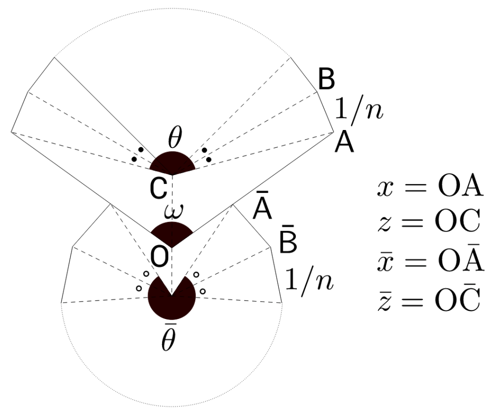

Figure 3: Shift of the enclosure's centre along the bisector of ω.

$$\begin{aligned} A_n(\omega;\theta) &= \frac{1}{4n}\cot\left(\frac{\theta}{2n}\right) + \frac{1}{4n^2}\sin^2\left(\frac{\theta}{2}\right)\operatorname{cosec}^2\left(\frac{\theta}{2n}\right)\left(\cot\left(\frac{\omega}{2}\right) - \cot\left(\frac{\theta}{2}\right)\right) \\ &\leq A_n^*(\omega) \end{aligned}$$

and

$$\begin{aligned} A_n^*(\omega) - A_n(\omega;\theta) = \frac{1}{4n}\frac{\sin\left(\frac{\omega}{2n}\right)}{\sin\left(\frac{\omega}{2}\right)}&\left(\cot\left(\frac{\omega}{2n}\right) - \cot\left(\frac{\theta}{2n}\right)\right) \\ &\times\left(\frac{\sin\left(\frac{\omega}{2}\right)}{\sin\left(\frac{\omega}{2n}\right)} - \frac{1}{n}\frac{\sin\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2n}\right)}\frac{\sin\left(\frac{\theta-\omega}{2}\right)}{\sin\left(\frac{\theta-\omega}{2n}\right)}\right). \end{aligned}$$

Over the other side of the walls, $(\bar{x}, \bar{z}) = (x, z) : (\omega, \theta) \mapsto (\bar{\omega}, \bar{\theta})$. Thus if $\theta + \omega \geq 2\pi$, then

$$A_n(\omega;\theta) + A_n(\bar{\omega};\theta) = \frac{1}{2n}\cot\left(\frac{\theta}{2n}\right) - \frac{1}{4n^2}\sin\theta\ \operatorname{cosec}^2\left(\frac{\theta}{2n}\right)$$

which is independent of ω.

The special case

$$\hat{\theta} = \frac{n(\omega+\pi)}{n+1}$$

has some appeal, as AC bisects $\angle OAB$. For this choice of θ we have

$$A_n(\omega;\hat{\theta}) = \frac{H_{2x}}{4n}\cot\left(\frac{\hat{\theta}}{2n}\right).$$

For example, $A_3\left(\frac{\pi}{2}, \frac{9\pi}{8}\right) = \frac{1}{12} + \frac{1}{8}\sqrt{4+2\sqrt{2}}$.
Sketch this enclosure with ruler and compass!