Parabola Volume 51, Issue 2 (2015)

Solutions 1471–1480

Q1471 Find the positive integer which has 7 proper divisors, with the sum of the proper divisors being 673. (Proper divisors are all divisors except the number itself: for example, the proper divisors of 20 are $1, 2, 4, 5, 10$.)

SOLUTION Suppose that a positive integer is written as a product of powers of s different prime numbers,

$$n = p_1^{a_1} p_2^{a_2} \cdots p_s^{a_s} .$$

Then every factor of n can be written in the form

$$p_1^{b_1} p_2^{b_2} \cdots p_s^{b_s} .$$

We must have $b_1 = 0$ or $b_1 = 1$ or $\ldots$ or $b_1 = a_1$, so there are $a_1 + 1$ possible values for b_1; a similar result holds for the other exponents; so the number of factors of n, including n itself, is

$$(a_1 + 1)(a_2 + 1) \cdots (a_s + 1) .$$

In this case we need $(a_1 + 1)(a_2 + 1) \cdots (a_s + 1) = 8$ and so there are three possibilities.

- Suppose that $s = 1$, so $a_1 = 7$ and $n = p_1^7$. Then the sum of the proper factors of n is

  $$1 + p_1 + p_1^2 + p_1^3 + p_1^4 + p_1^5 + p_1^6 ,$$

  and it is easy to check that this cannot equal 673: if $p_1 = 2$ then the sum is 127, and if $p_1 \geq 3$ then the sum is too big.

- Next try $s = 2$, $a_1 = 3$, $a_2 = 1$, so that $n = p^3 q$. (We have written p, q instead of p_1, p_2.) One of the proper factors of n is p^3, so we have $p^3 < 673$ and hence $p < 9$; moreover, the sum of all proper factors is

  $$1 + p + p^2 + p^3 + q + pq + p^2 q = 673$$

  and therefore

  $$q = \frac{673 - 1 - p - p^2 - p^3}{1 + p + p^2} .$$

  Since p must be prime we have the possibilities $p = 2, 3, 5, 7$ giving respectively

  $$q = 94 \, , \ \frac{633}{13} \, , \ \frac{517}{31} \, , \ \frac{91}{19}$$

  which are not allowable (remember that q must be prime).

- The final possibility is that $s = 3$ and $a_1 = a_2 = a_3 = 1$, so that $n = pqr$; we may as well assume that $p < q < r$. Adding up all the proper factors of n gives

$$1 + p + q + r + pq + pr + qr = 673\ . \qquad (*)$$

Now p cannot be 2 as then q, r would be odd and the left-hand side would be even. Therefore $q \geq p + 2$ and $r \geq p + 4$, and the previous equation implies

$$1 + p + (p + 2) + (p + 4) + p(p + 2) + p(p + 4) + (p + 2)(p + 4) \leq 673\ .$$

This simplifies to

$$3p^2 + 15p + 15 \leq 673$$

and so $p < 13$; since p is prime, we have $p = 3, 5, 7$ or 11. Substituting $p = 11$ into $(*)$ and using the fact that $r \geq q + 2$ gives

$$12 + q + (q + 2) + 11q + 11(q + 2) + q(q + 2) \leq 673\ ;$$

this leads to $q < 16$, so $q = 13$ and

$$r = \frac{673 - 1 - p - q - pq}{1 + p + q} = \frac{101}{5}$$

which is not possible. By similar calculations we can rule out all possibilities except for $p = 5$, $q = 13$, $r = 31$, so...

the only possible answer is $n = 2015$. (Surprise?)

Q1472 Let m, n be positive integers, and define a sequence of numbers $T_1, T_2, T_3, \ldots$ by the following rules: $T_1 = -2015$, $T_2 = 2015$ and

$$T_k = \frac{mT_{k-2} + nT_{k-1}}{m + n} \qquad (*)$$

for $k \geq 3$. If the value of T_k gets closer and closer to 1729 as k increases, find the ratio m/n.

SOLUTION First we observe that $(*)$ implies

$$\begin{aligned} T_k - T_{k-1} &= \frac{mT_{k-2} + nT_{k-1}}{m + n} - T_{k-1} \\ &= \frac{mT_{k-2} - mT_{k-1}}{m + n} \\ &= \Big(\frac{-m}{m + n}\Big)(T_{k-1} - T_{k-2}) \end{aligned}$$

for $k \geq 3$. Now

$$T_k - T_1 = (T_2 - T_1) + (T_3 - T_2) + (T_4 - T_3) + \cdots + (T_k - T_{k-1})\ ,$$

and from what we have just proved, the right-hand side is a geometric series with common ratio $-m/(m+n)$. Since m and n are positive, this ratio has absolute value less than 1; so $T_k - T_1$ gets closer and closer to the infinite sum

$$(T_2 - T_1) + \left(\frac{-m}{m+n}\right)(T_2 - T_1) + \left(\frac{-m}{m+n}\right)^2 (T_2 - T_1) + \cdots = \frac{T_2 - T_1}{1 + \dfrac{m}{m+n}}$$

and T_k gets closer to

$$\frac{T_2 - T_1}{1 + \dfrac{m}{m+n}} + T_1 \, .$$

Using the given information,

$$\frac{4030}{1 + \dfrac{m}{m+n}} - 2015 = 1729 \, ,$$

which gives

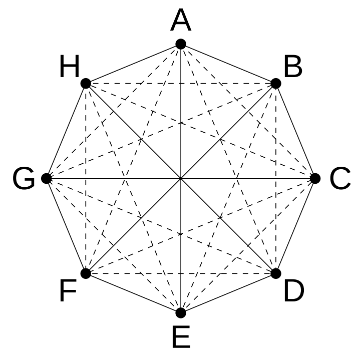
$$\frac{m}{m+n} = \frac{4030}{3744} - 1 = \frac{11}{144}$$

and finally

$$\frac{m}{n} = \frac{1}{\dfrac{m+n}{m} - 1} = \frac{11}{133} \, .$$

Q1473 In Problem 1465 (solution previous issue) we proved that among any nine people, there *must* always be four who pairwise like each other or three who pairwise dislike each other. Prove that this is not true for eight people: that is, it is possible to have a group of eight people in which no four pairwise like each other, and no three pairwise dislike each other.

SOLUTION We wish to draw a diagram consisting of eight points, with a white line joining two points if the corresponding people like each other, and a black line if they do not, in such a way that there are no four points joined exclusively by white lines, and no three points joined exclusively by black points. From the solution to Problem 1465, we know that this is impossible if any of the points is attached to four or more black lines, so we shall try to give each point just three black lines. After a bit of trial and error we may come up with the following diagram (solid lines are black, dotted lines are white).



There are no three points joined by black lines: to see this, notice that the arrangement of colours around each of the eight points is exactly the same and so we only need to look at one point, say A. This point is joined by black lines to B, E and H; no two of these are joined by black lines. Similarly, if four points are all joined by white lines then we may assume that one of the points is A and the others are three of C, D, F, G; if C is omitted then A, D, F, G are not joined by a white line from F to G; likewise if D, F or G is omitted.

Therefore, in this arrangement of likes and dislikes among eight people, no four pairwise like each other, and no three pairwise dislike each other.

Q1474 Consider the equation

$$\lfloor 1000000\sqrt{m} \rfloor = \lfloor 1000000\sqrt{n} \rfloor ,$$

where m and n are to be integers. We have shown (Problem 1467, solution last issue) that a possible solution is $m = 500000^2$ and $n = 500000^2 + 1$. Prove that there is in fact no smaller integer value of m for which

$$\lfloor 1000000\sqrt{m} \rfloor = \lfloor 1000000\sqrt{m+1} \rfloor .$$

SOLUTION If $1000000\sqrt{m}$ and $1000000\sqrt{m+1}$ give the same result when rounded to the nearest integer downwards, their difference must be less than 1: that is,

$$1000000\sqrt{m+1} - 1000000\sqrt{m} < 1 .$$

But this implies

$$\begin{aligned} 1 &> 1000000\big(\sqrt{m+1} - \sqrt{m}\big) \\ &= 1000000\frac{\big(\sqrt{m+1} - \sqrt{m}\big)\big(\sqrt{m+1} + \sqrt{m}\big)}{\sqrt{m+1} + \sqrt{m}} \\ &= \frac{1000000}{\sqrt{m+1} + \sqrt{m}} \\ &> \frac{1000000}{2\sqrt{m+1}} \end{aligned}$$

and so

$$m + 1 > 500000^2 .$$

Hence, as required, no value of m smaller than 500000^2 is possible.

Q1475 Prove that if

$$x \sin\Big(\theta - \frac{2\pi}{3}\Big) = y \sin\theta = z \sin\Big(\theta + \frac{2\pi}{3}\Big) \qquad (*)$$

then

$$xy + yz + zx = 0 .$$

SOLUTION First notice that $\theta - \frac{2\pi}{3}$ and θ and $\theta + \frac{2\pi}{3}$ do not differ by integer multiples of π; therefore only one, if any, of $\sin\big(\theta - \frac{2\pi}{3}\big)$ and $\sin\theta$ and $\sin\big(\theta + \frac{2\pi}{3}\big)$ can be zero. If $\sin\theta$ is zero and the others are not, then $(*)$ shows that $x = z = 0$ and hence $xy + yz + zx = 0$. A similar argument holds if either of the other sine terms is zero.

So now we can consider the case when none of the sine terms is zero. Write

$$K = x \sin\Big(\theta - \frac{2\pi}{3}\Big) = y \sin\theta = z \sin\Big(\theta + \frac{2\pi}{3}\Big) ;$$

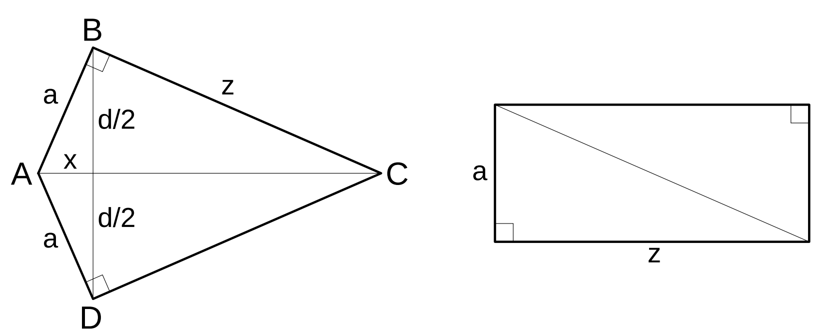
then

$$\begin{aligned}
(xy + yz + zx) \sin\Big(\theta - \frac{2\pi}{3}\Big) \sin(\theta) \sin\Big(\theta + \frac{2\pi}{3}\Big) \\
&= K^2 \Big(\sin\Big(\theta + \frac{2\pi}{3}\Big) + \sin\Big(\theta - \frac{2\pi}{3}\Big) + \sin\theta\Big) \\
&= K^2 \Big(2 \sin\theta \cos\frac{2\pi}{3} + \sin\theta\Big) ,
\end{aligned}$$

where the last step uses standard trigonometric formulae. But since $\cos\frac{2\pi}{3} = -\frac{1}{2}$, this is zero; and since none of the sine factors is zero, we have $xy + yz + zx = 0$.

Q1476 In quadrilateral $ABCD$, the angles at B and D are right angles, and $AB = AD$. Find the area of the quadrilateral in terms of the lengths $a = AB$ and $d = BD$.

SOLUTION From the diagram, it is clear that the quadrilateral can be cut into two triangles and rearranged so as to form a rectangle with area az. So our problem is to find z in terms of a and d.



By similar triangles and Pythagoras we have

$$\frac{x}{a} = \frac{d/2}{z} \quad \text{and} \quad x^2 + \left(\frac{d}{2}\right)^2 = a^2 \ ;$$

eliminating x from these equations, we have

$$z^2 = \frac{a^2d^2}{4x^2} = \frac{a^2d^2}{4a^2 - d^2} \ .$$

So the area of the quadrilateral is

$$az = \frac{a^2d}{\sqrt{4a^2 - d^2}} \ .$$

Q1477 A circus is organising a touring schedule under the following conditions. There are k performances to be arranged; there are m possible venues, and n possible performance days. No two performances are to be given at the same venue, and no two performances are to be given on the same day. In how many ways can the tour be scheduled? (Ignore any difficulties connected with the time needed for setting up, packing up and travelling!)

SOLUTION First choose k of the possible m venues; at this stage, order is not important, so the number of choices is $C(m, k)$. Next choose k of the n possible dates: the number of choices is $C(n, k)$. Finally, place the k chosen venues in order, so as to decide which is used on which date: this can be done in $k!$ ways. The total number of possible schedules is

$$C(m, k)C(n, k)k! \ .$$

Note: $C(m, k)$ is the **combination symbol** or **binomial coefficient** which you may have seen written as ${}^mC_k$ or $\binom{m}{k}$.

Q1478 Mitchell and Dale are now playing a game with ordinary (six–sided) dice. The rules are as follows: if a player throws a 6 then he wins; if he throws a 1 or a 2 then he loses; if he throws a $3, 4$ or 5 then the other player has a turn. What is the first player's chance of winning?

SOLUTION We follow the method of solution of Problem 1457 from *Parabola* Volume 50, Issue 3. Let p be the first player's probability of winning overall. His chance of winning on his first throw is $\frac{1}{6}$. If he does not win on his first throw, there are two options which will still allow him to win. He could pass the die to the other player who could then make a losing throw: the probability of this is $(\frac{3}{6})(\frac{2}{6}) = \frac{1}{6}$. Alternatively, he could pass the die and the second player could pass it back: probability $(\frac{3}{6})(\frac{3}{6}) = \frac{1}{4}$. The first player is then exactly in the same position as he was at the beginning of the game and his winning probability from here is just p. Putting all this together gives the equation

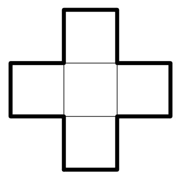
$$p = \frac{1}{6} + \frac{1}{6} + \frac{1}{4}p \ ,$$

which can be solved to give the winning probability

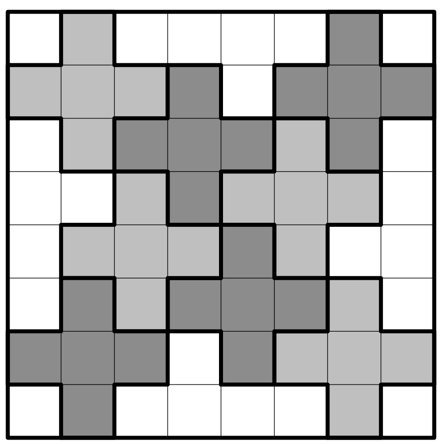
$$p = \frac{4}{9} .$$

Check: Since the first player has a higher chance of losing than winning on the first throw of the game, it makes sense that his overall winning probability should work out as less than $\frac{1}{2}$.

Q1479 Determine the maximum number of cross shapes (as in the diagram) that can be placed on an 8×8 chessboard. Each cross must cover exactly five squares on the board, and crosses may not overlap. Prove that the number you claim can be placed as required, and that more than this many cannot.



SOLUTION The maximum number of crosses that can be placed on the board without overlapping is 8. It is not hard to find an arrangement of eight crosses by trial and error: the diagram shows one of many possibilities.



Now we have to prove that it is not possible to place nine crosses on the board without overlapping. First note that it is not possible for any part of a cross to cover a corner square of the board without the cross sticking out beyond the edges. Next, consider the six squares, other than the corner squares, on each side of the board. It is not hard to convince yourself that crosses (not extending beyond the board) can only cover a maximum of two squares out of these six. Therefore there must be at least four uncovered squares on each side, sixteen altogether; plus the four corners, making at least 20 uncovered squares. So the crosses cover no more than 44 squares, and as each occupies 5 squares, there is not room enough for 9 crosses.

NOW TRY problem 1486.

Q1480 Consider the number

$$S = \frac{2}{101} + \frac{4}{10001} + \frac{8}{100000001} + \cdots + \frac{1024}{1000\cdots0001}\,,$$

where the numerators are powers of 2, and the number of zeros in each denominator is always one less than the corresponding numerator. Write S as an infinite recurring decimal.

SOLUTION The kth term on the right-hand side is

$$\frac{2^k}{10^{2^k}+1}\,.$$

Now observe that

$$\frac{2^{k+1}}{10^{2^{k+1}}-1} = \frac{2^k[(10^{2^k}+1)-(10^{2^k}-1)]}{(10^{2^k}+1)(10^{2^k}-1)} = \frac{2^k}{10^{2^k}-1} - \frac{2^k}{10^{2^k}+1}\,;$$

therefore the kth term can be written alternatively as

$$\frac{2^k}{10^{2^k}-1} - \frac{2^{k+1}}{10^{2^{k+1}}-1}\,.$$

So the entire series is equal to

$$\begin{aligned}\Big(\frac{2}{10^2-1} - \frac{4}{10^4-1}\Big) + \Big(&\frac{4}{10^4-1} - \frac{8}{10^8-1}\Big)\\ &+ \Big(\frac{8}{10^8-1} - \frac{16}{10^{16}-1}\Big)\\ &+ \cdots\\ &+ \Big(\frac{512}{10^{512}-1} - \frac{1024}{10^{1024}-1}\Big)\\ &+ \Big(\frac{1024}{10^{1024}-1} - \frac{2048}{10^{2048}-1}\Big)\,.\end{aligned}$$

All of the terms except the first and last cancel to give

$$S = \frac{2}{10^2-1} - \frac{2048}{10^{2048}-1} = \frac{2}{99} - \frac{2048}{999\ldots999}\,,$$

where there are 2048 nines in the last denominator. Now if the denominator of a fraction $p/q < 1$ is the number consisting of n digits, all equal to 9, then p/q can be written as a repeating decimal in which the repeating part has length n and contains the digits of p, preceded by a sufficient number of 0s to give that length. (We ask you to prove this in Problem 1481 in this issue.) Thus

$$\frac{2}{99} = 0\cdot02020202020202\ldots\,,$$

and

$$\frac{2048}{10^{2048}-1} = 0 \cdot 000\ldots0002048\ldots\ ,$$

where there are 2044 zeros and the decimal now repeats. Subtracting these gives

$$S = 0 \cdot 020202\ldots020202018154\ldots\ ,$$

where the pair 02 occurs 1021 times, and the entire part after the decimal point which we have shown now repeats.

NOW TRY problem 1481.