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Counting paths on grids with obstructions

Anthony Wang[1]

Abstract

This article describes how to count the number of paths across an obstructed grid. The problem's fundamental connection to the Inclusion-Exclusion Principle is illustrated, and intriguing relationship to Pascal's Triangle is expanded upon.

1 Introduction

One of the applications of Pascal's Triangle concerns the number of possible paths on a grid. A traditional version of this problem asks to find the number of ways to move from the top-left corner (S) to the bottom-right corner (F) of the grid, following the grid lines only in the downward and rightward directions (see Figures 1 and 3).

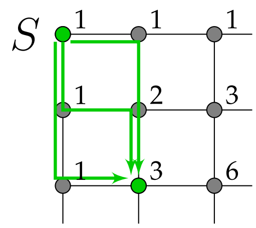

Figure 1: Possible paths from S to a point on the grid

Mohaty provides a detailed exploration of many theorems related to path counting [2]; meanwhile, study of the problem has also been expanded to three-dimensional grids [1]. Here, additional parameters are introduced to the traditional problem in the form of obstructions positioned between certain adjacent junctions. Given n obstructions placed at location(s) on the grid, how do the obstructions together affect the number of possible paths F_n? With an expository dissection of the problem, this article models the scenario in general terms while revealing the interesting, hidden-in-plain-sight connections to Pascal's Triangle and the Inclusion-Exclusion Principle.

1.1 Distinct paths

Between the starting point S at the top left-corner and any other junction on the grid, there are certain possible paths connecting the two points. These paths each consist of a uniquely-ordered sequence of horizontal and vertical moves. Figure 1 shows an example of the three possible paths from S to the other highlighted point.

[1]Anthony Wang is a high school student at The Woodlands School in Mississauga ON, who will be studying computer science at Rice University.

2 Understanding various obstruction cases

2.1 Grids with no obstruction

In the case of an unobstructed grid, mechanically recording the number of possible paths at each junction creates a cross-section of a Pascal's Triangle.

$$\begin{array}{ccccccccccc} &&&&&1&&&&& \\ &&&&1&&1&&&& \\ &&&1&&2&&1&&& \\ &&1&&3&&3&&1&& \\ &1&&4&&6&&4&&1& \\ 1&&5&&10&&10&&5&&1 \end{array} \qquad \begin{array}{ccccccccccc} &&&&&\binom{0}{0}&&&&& \\ &&&&\binom{1}{0}&&\binom{1}{1}&&&& \\ &&&\binom{2}{0}&&\binom{2}{1}&&\binom{2}{2}&&& \\ &&\binom{3}{0}&&\binom{3}{1}&&\binom{3}{2}&&\binom{3}{3}&& \\ &\binom{4}{0}&&\binom{4}{1}&&\binom{4}{2}&&\binom{4}{3}&&\binom{4}{4}& \\ \binom{5}{0}&&\binom{5}{1}&&\binom{5}{2}&&\binom{5}{3}&&\binom{5}{4}&&\binom{5}{5} \end{array}$$

Figure 2: Two representations of Pascal's Triangle, rows 0 to 5

Figure 2 illustrates separately the first several rows of Pascal's Triangle, expressed in numerical and combinatorial forms. A term in the nth row and kth column is given by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

where n and k are integers satisfying $0 \leq k \leq n$. Each entry is the sum of the two entries above it in the previous row:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

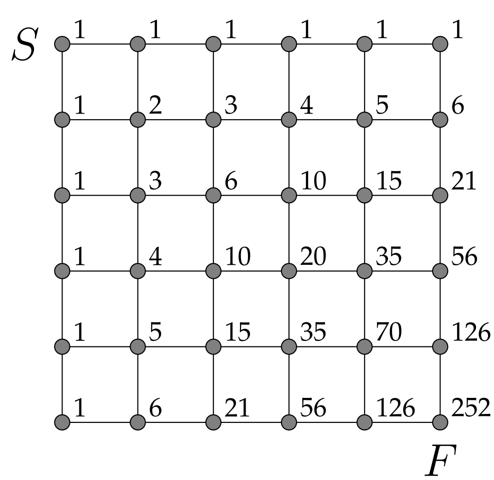

Figure 3: Grid with no obstruction

On the 5×5 grid in Figure 3, the number of paths from S to F lies in the 10th row and 5th column of Pascal's Triangle and can therefore be written as

$$F_0 = 252 = \binom{10}{5}.$$

One may also arrive at this by noting that 10 moves must be made from S to F, of which 5 must be horizontal and 5 must be vertical. In the case of a rectangular 4×5 grid,

$$F_0 = 126 = \binom{9}{4} = \binom{9}{5}.$$

As Mohaty notes as part of his larger work on path counting [2], this may be generalized for any $\ell \times w$ size grid as either

$$F_0 = \binom{\ell + w}{\ell} \quad \text{or} \quad F_0 = \binom{\ell + w}{w}.$$

Introducing n obstructions will decrease the number of possible paths by some number N:

$$F_n = F_0 - N.$$

By determining the value of N, we can determine the value of F_n.

2.2 One obstruction

The first cases with obstructions can again be approached mechanically, by cumulatively recording the number of possible paths at each junction. However, an intriguing insight is highlighted in Figure 4, where the obstruction is positioned after the first junction.

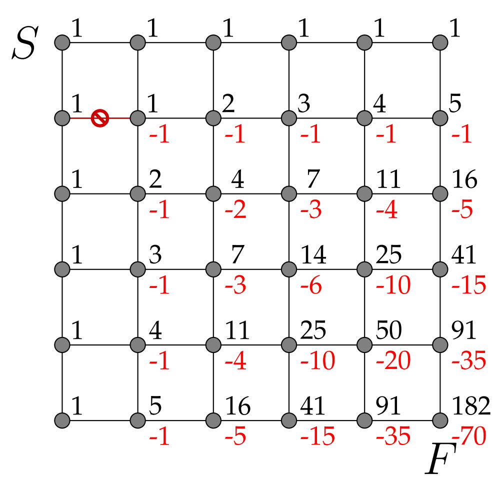

Figure 4: One obstruction

The number of possible paths to each subsequent junction is decreased by a certain number relative to the unobstructed case. These differences are shown in red, and reveal yet another Pascal's Triangle. Since it signifies the reduction in the number of paths to a junction, this new triangle can be referred to as the *relative triangle*.

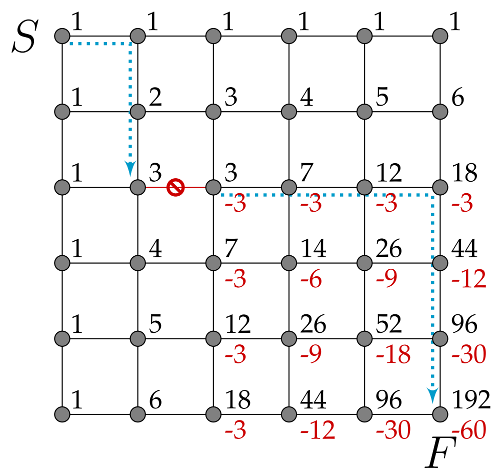

Figure 5: One obstruction with $k = 3$

However, in Figure 5 the obstruction has been moved further from the start, and the relative triangle at first no longer seems to exist – but a form of it does in fact remain; the values of the relative triangle have simply been multiplied by a common coefficient k. Moreover, this value k is the number of possible paths to the junction where the obstruction begins. Therefore, compared to an unobstructed grid, the number of possible paths at F will be less by kr, where r is the number of paths from the junction immediately after the obstruction to the end of the grid. Therefore, for a single obstruction, the number of paths from S to F is

$$F_1 = F_0 - kr$$

which, as the arrows in Figure 5 demonstrate, also equal to

$$F_1 = \binom{l+w}{l} - \binom{l_k + w_k}{l_k}\binom{l_r + w_r}{l_r}.$$

2.3 A remark on notation

To explore more complex cases, we will need some more notation. Continuing the iterative approach described above will at first seem to greatly increase the complexity of the equations – but it will ultimately lend itself well to a pleasing insight.

When there are multiple obstructions, the number of paths up to the point where the nth obstruction begins will be designated k_n. The use of this notation assumes that there is no dependency on previous obstructions. Instead, let a different coefficient Q_n take this dependency into account. This number Q_n will be referred to as the *true*

Relative Triangle Coefficient (RTC). This distinction is important because although $Q_1 = k_1$, this does not hold for any greater value of n. Meanwhile, the number of paths from the point immediately after the obstruction to the end of the grid, once again ignoring any further obstructions, will be denoted by r_n.

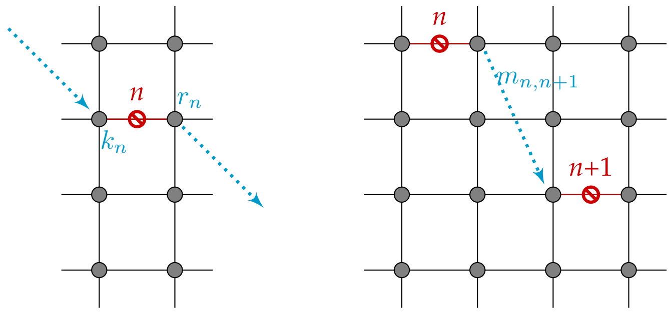

Figure 6: Notation

For multiple obstructions, we must count the paths between two successive obstructions: the number of paths between the nth and $(n + 1)$th obstructions will be denoted by $m_{n,n+1}$. For example, the number of paths between the first and second obstructions is $m_{1,2}$. These numbers are shown in Figure 6.

2.4 Two or three obstructions

For two obstructions, the path numbers must first be calculated for the first obstruction only, and then an additional number must be subtracted for the second obstruction:

$$F_2 = F_1 - Q_2 r_2.$$

Additionally, the RTC (Q-value) for the second obstruction has a dependency on the first obstruction. Therefore, Q_2 must be found by considering the lower-right subgrid with only the first obstruction present:

$$Q_2 = k_2 - k_1 m_{1,2}.$$

Since $F_1 = F_0 - k_1 r_1$,

$$F_2 = F_0 - k_1 r_1 - (k_2 - k_1 m_{1,2}) r_2.$$

The approach when there are three obstructions is much the same. However, each added obstruction makes solutions considerably more difficult, as the number of dependencies grows:

$$\begin{aligned} F_3 &= F_2 - Q_3 r_3 \\ &= F_0 - k_1 r_1 - (k_2 - k_1 m_{1,2}) r_2 - \big(k_3 - k_1 m_{1,3} - (k_2 - k_1 m_{1,2}) m_{2,3}\big) r_3. \end{aligned}$$

3 The recursive nature of counting paths

Clearly, this approach becomes tedious as more obstructions are added. This is partly because the problem is recursive: to resolve any grid with n obstructions, the same grid must first be solved for the case in which there are $n-1$ obstructions, and so on. In addition, the RTC for the nth obstruction is dependent on that of the $(n-1)$th obstruction, and so on likewise. Indeed, inspection of the four cases above reveals the recursive sequence

$$F_n = F_{n-1} - Q_n r_n$$

with Q_n itself also growing recursively. A different approach is therefore required to illuminate the solution.

4 The Inclusion-Exclusion Principle

In their recursive form, the equations created to model this problem do not at first show a clear relation to the Inclusion-Exclusion Principle. However, it would be reasonable to expect it to apply here, for reasons that will be described shortly.

4.1 Background

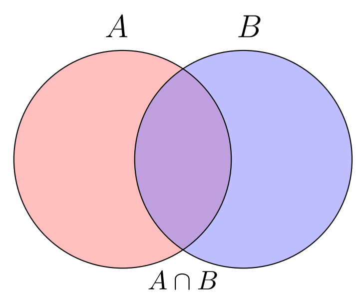

Figure 7: Two sets

The Inclusion-Exclusion Principle is the observation that the number of elements in the union of several sets can be found by adding the number of elements in each and then, to avoid double counting, subtracting the number of elements in each intersection of sets. For two sets, as shown in Figure 7, the Principle is expressed symbolically as

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

For three sets, as shown in Figure 8, the Principle appears as

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

Note that in this case, it is necessary to add back the triple-wise intersection since too much was deleted when intersection sizes were subtracted.

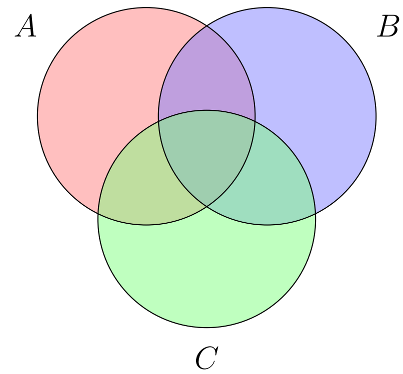

Figure 8: Three sets

Figure 9 gives a numerical example for three overlapping sets. The cardinalities of each subset are shown in Table 1. The number of elements in the union of sets A, B, and C will be less than the simple sum of $|A| + |B| + |C|$, since the overlapping subsets must be accounted for. Indeed,

$$|A \cup B \cup C| = 30 + 27 + 24 - 17 - 9 - 9 + 3 = 49.$$

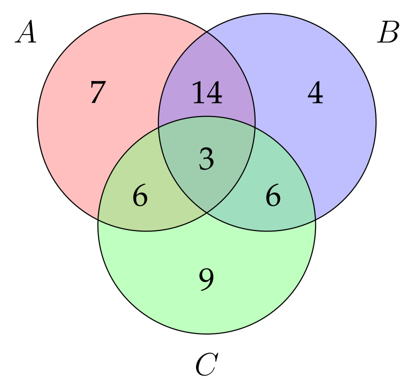

Figure 9: A numerical example of three sets

Set	Cardinality
A	30
B	27
C	24
$A \cap B$	17
$A \cap C$	9
$B \cap C$	9
$A \cap B \cap C$	3

Table 1: Subset cardinalities for Figure 9

4.2 Encountering the Principle

Returning to the problem at hand, a satisfying connection soon reveals itself. Take, for example, the equation for a two-obstruction case

$$F_2 = F_0 - k_1r_1 - (k_2 - k_1m_{1,2})r_2 = F_0 - (k_1r_1 + k_2r_2 - k_1m_{1,2}r_2).$$

The connection in question becomes clearer still with the expansion of F_3:

$$\begin{aligned}F_3 &= F_0 - k_1r_1 - k_2r_2 - k_3r_3 + k_1m_{1,2}r_2 + k_1m_{1,3}r_3 + k_2m_{2,3}r_3 - k_1m_{1,2}m_{2,3}r_3 \\ &= F_0 - (k_1r_1 + k_2r_2 + k_3r_3 - k_1m_{1,2}r_2 - k_1m_{1,3}r_3 - k_2m_{2,3}r_3 + k_1m_{1,2}m_{2,3}r_3).\end{aligned}$$

A familiar pattern is emerging! Recall that each term k_ir_i represents the number of paths "killed off" by the ith obstruction, if that obstruction were considered in isolation. Let N_i be the set of all those paths; then $|N_i| = k_ir_i$. Table 2 relates these subsets to the terms representing their cardinalities for the three-obstruction case.

Set	Cardinality
N_1	k_1r_1
N_2	k_2r_2
N_3	k_3r_3
$N_1 \cap N_2$	$k_1m_{1,2}r_2$
$N_1 \cap N_3$	$k_1m_{1,3}r_3$
$N_2 \cap N_3$	$k_2m_{2,3}r_3$
$N_1 \cap N_2 \cap N_3$	$k_1m_{1,2}m_{2,3}r_3$

Table 2: Subset cardinalities for grids with three obstructions

The expansions of F_2 and F_3 can therefore also be written as:

$$\begin{aligned}F_2 &= F_0 - (|N_1| + |N_2| - |N_1 \cap N_2|) \\ F_3 &= F_0 - (|N_1| + |N_2| + |N_3| - |N_1 \cap N_2| - |N_1 \cap N_3| - |N_2 \cap N_3| + |N_1 \cap N_2 \cap N_3|.\end{aligned}$$

Expanded, it is evident that N, the number of paths subtracted from F_0, is simply

$$|N_1 \cup N_2 \cup \cdots \cup N_n|.$$

The Principle should indeed apply here because to solve the problem, one must find the paths affected by each blockage, but without over-counting the "overlapping" sets of paths (those that are prohibited by two or more obstructions). The analysis above confirms why this is true. The number of possible paths on a grid with n obstructions can now be written as explicit formulas, using the Inclusion-Exclusion Principle:

$$F_n = \binom{l+w}{l} - \left|\bigcup_{i=1}^{n} N_i\right| = \binom{l+w}{l} - \sum_{i=1}^{n} |N_i| + \sum_{1 \le i < j \le n} |N_i \cap N_j| - \cdots$$

where

$$|N_i| = k_ir_i = \binom{l_{k_i} + w_{k_i}}{l_{k_i}}\binom{l_{r_i} + w_{r_i}}{l_{r_i}}, \qquad |N_i \cap N_j| = k_im_{i,j}r_j, \qquad \text{and so on.}$$

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References

[1] J.F. Lucas, Paths and Pascal numbers, *The Two-Year College Mathematics Journal* **14**(4) (1983), 329–341.

[2] G. Mohaty, *Lattice Path Counting and Applications*, Academic Press, Inc., New York, NY, 1979.