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The tangency equivalence class and indirect differentiation

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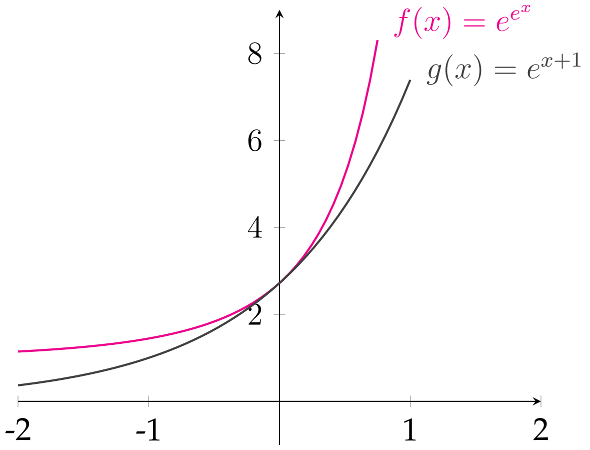
1 Introduction

It is very interesting how mathematical notation can serve as a powerful tool for solving or simplifying solutions to various problems. "A good notation may be wonderfully suggestive, leading to real conceptual breakthroughs" [1]. A widely-known example of this is the relation of congruence. One cannot help but wonder how the mere act of giving a symbol to the sentence "a and b have the same remainder upon division by n" leads to insights so significant that it is considered a fundamental pillar to number theory.

This observation might hopefully stimulate in the reader's mind a desire to invent their own mathematical relation and investigate its properties. This reasoning underpins the purpose of this article which is to introduce a useful notation for equating the tangents of functions.

2 A relation on functions

Consider the functions $f(x) = e^{e^x}$ and $g(x) = e^{x+1}$ shown below.



Upon graphing these functions, one would notice that they appear tangent at $x = 0$. This is verifiable; their value and derivative at the relevant input are identical, so their

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[1] Benson is an electrical engineering student with an unhealthy obsession with maths and who should spend more time reading about engineering instead.

tangent lines coincide. However, it is possible to intuitively understand this event of tangency: it is famously known that the line tangent to e^x at $x = 0$ is $x + 1$. These latter two functions are the exponents of f and g. It's as if the exponentiation of tangent functions yields functions that are still tangent. That makes sense: if k and h have close values around x_0, then e^k and e^h should also be close around x_0.

This shouldn't be a quality of exponential functions only. Squaring tangent functions should also preserve tangency. If k is tangent to h at x_0, then $k(x) + 1$ should be tangent to $h(x)+1$ at x_0. These meditations, together with the fact that it's cumbersome to keep writing "is tangent to" each time the above reasoning is invoked, lead naturally to the invention of notation that abbreviates and simplifies handling tangency. To this end, the following definition is presented.

Definition 1. *Consider the family $\mathcal{F}$ of real functions that are continuous and differentiable at $x = x_0$ for some real number x_0. The relation $\dashv$ on $\mathcal{F}$ with respect to x_0 is defined as follows. For each $f, g \in \mathcal{F}$,*

$$[f \dashv g][x_0] \quad \textit{if and only if} \quad f(x_0) = g(x_0) \quad \textit{and} \quad f'(x_0) = g'(x_0).$$

If x_0 is clearly given from the context, then $[f \dashv g][x_0]$ may be written briefly as $f \dashv g$.

In other words, $[f \dashv g][x_0]$, or $f \dashv g$, means "f is tangent to g at $x = x_0$". It is easy to see that this relation is reflexive, symmetric and transitive:

Proposition 2. *For all real functions f, g and h that are differentiable at x_0,*

- *(R)* $f \dashv f$;
- *(S)* *if* $f \dashv g$, *then* $g \dashv f$;
- *(T)* *if* $f \dashv g$ *and* $g \dashv k$, *then* $f \dashv k$.

The next natural step is to figure out the properties of this tangency relation. Here are a few of such properties.

Proposition 3. *Let f, g, h, k be real functions that are differentiable at x_0, and T a real function differentiable at $x = f(x_0)$. Then*

- *(a)* *if* $f \dashv g$, *then* $f + h \dashv g + h$;
- *(b)* *if* $f \dashv g$, *then* $fh \dashv gh$;
- *(c)* *if* $f \dashv g$, *then* $T \circ f \dashv T \circ g$.
- *(d)* *if* $f \dashv g$ *and* $h \dashv k$, *then* $f + h \dashv g + k$;
- *(e)* *if* $f \dashv g$ *and* $h \dashv k$, *then* $fh \dashv gk$.

In part (c), $T \circ f$ and $T \circ g$ denote the functions $T(f(x))$ and $T(g(x))$, respectively.

Proof. To prove (d), suppose that $f \dashv g$ and $h \dashv k$. Then

$$\begin{array}{llll} f(x_0) = g(x_0) & \text{and} & f'(x_0) = g'(x_0) & \text{and} \\ h(x_0) = k(x_0) & \text{and} & h'(x_0) = k'(x_0)\,, & \end{array}$$

so

$$\begin{aligned} (f+h)(x_0) &= f(x_0) + h(x_0) = g(x_0) + k(x_0) = (g+k)(x_0) \quad \text{and} \\ (f+h)'(x_0) &= f'(x_0) + h'(x_0) = g'(x_0) + k'(x_0) = (g+k)'(x_0)\,. \end{aligned}$$

In other words, $f + h \dashv g + k$, which is what we wanted to prove.

Part (e) is proved similarly. Parts (a) and (b) follow from (d) and (e), respectively, since $h \dashv h$ by Proposition 2.

Finally, to prove (c), suppose that $f \dashv g$. Then $f(x_0) = g(x_0)$ and $f'(x_0) = g'(x_0)$, so

$$\begin{aligned} & (T \circ f)(x_0) = T(f(x_0)) = T(g(x_0)) = (T \circ g)(x_0) \\ \text{and} \quad & (T \circ f)'(x_0) = f'(x_0)T'(f(x_0)) = g'(x_0)T'(g(x_0)) = (T \circ f)'(x_0)\,. \end{aligned}$$

That is, $T \circ f \dashv T \circ g$, which is what we wanted to prove. □

Proposition 4. *Let f and g be real functions that are differentiable at x_0, and let a and b be real numbers where $a \neq 0$. If $[f(x) \dashv g(x)][x_0]$, then $[f(ax+b) \dashv g(ax+b)]\left[\frac{x_0-b}{a}\right]$.*

The intuitive meaning of this proposition will be explained below, and the proof is left as an exercise for the reader.

Letting $a = 1$, Proposition 4 is reduced to stating that if $[f(x) \dashv g(x)][x_0]$, then $[f(x+b) \dashv g(x+b)][x_0 - b]$. This says that when f and g are both tangent at x_0, and you shift both to the left by the same amount b, the resulting functions are still tangent at $x = x_0 - b$.

Letting $b = 0$, Proposition 4 is reduced to stating that if $[f(x) \dashv g(x)][x_0]$, then $[f(ax) \dashv g(ax)]\left[\frac{x_0}{a}\right]$. This says that when f and g are both tangent at x_0, compressing the functions by the same amount a results in functions that are still tangent at $x = \frac{x_0}{a}$.

3 Applications

Applications of the relation $\dashv$ and its properties above are illustrated below.

Problem 1. *Show that, for each real number m, the functions x^2 and $2mx - m^2$ are tangent at $x = m$.*

Solution. Since $[x^2 \dashv 0][0]$, Proposition 4 implies that $\left[(x-m)^2 \dashv 0\right][m]$. Therefore,

$$[x^2 - 2mx + m^2 \dashv 0][m]\,.$$

Proposition 3(a) allows us to add $2mx - m^2$ to both sides:

$$[x^2 \dashv 2mx - m^2][m]\,.$$

In other words, x^2 and $2mx - m^2$ are tangent at $x = m$. □

Problem 2. *Show that, for each real number m, the natural logarithm $\ln x$ has the tangent $y = \frac{x}{m} - 1$ at $x = m$.*

Solution. First note that $[\ln(x) \dashv x - 1][1]$. By Proposition 4, $\left[\ln\left(\frac{x}{m}\right) \dashv \frac{x}{m} - 1\right][m]$, so

$$\left[\ln(x) - \ln(m) \dashv \frac{x}{m} - 1\right][m]\,.$$

By Proposition 3(a),

$$\left[\ln(x) \dashv \frac{x}{m} - 1 + \ln(m)\right][m]\,,$$

which is what we wanted to show.

Problem 3. *Find an exponential function ab^x that is tangent to $\cos(x)$ at $x = m$.*

Solution. The way this problem would be naturally approached is to find a and b such that $f(m) = g(m)$ and $f'(m) = g'(m)$ where $f(x) = \cos(m)$ and $g(x) = ab^m$. Solving such system of equations can be annoying and time consuming. However, by using our tangency relation and its properties, this problem can be solved simply.

First note that $(\ln(\cos(x)))' = -\frac{\sin(x)}{\cos(x)} = -\tan(x)$. The function $\ln(\cos(x))$ therefore has tangent line $-\tan(m)(x - m) + \ln(\cos(m))$ at $x = m$, so

$$\left[\ln(\cos(x)) \dashv -\tan(m)(x - m) + \ln(\cos(m))\right][m]\,.$$

By Proposition 3(c), exponentiation of both sides yields

$$\left[\cos(x) \dashv \cos(m)e^{-\tan(m)(x-m)}\right][m]\,.$$

Therefore, setting $a = \cos(m)e^{m\tan(m)}$ and $b = e^{-\tan(m)}$ gives an exponential function ab^x that is tangent to $\cos(x)$ at $x = m$. $\square$

Problem 4. *Find the derivative of $f(x) = e^{e^{e^x}}$ at $x = 0$.*

Solution. The consecutive application of nested chain rules would solve this problem but the following approach is easier.

First, note that $[e^x \dashv x + 1][0]$. By Proposition 3(c), taking the natural logarithm yields $[x \dashv \ln(x + 1)][0]$, and so $[x + 1 \dashv \ln(x + 1) + 1][0]$ by Proposition 3(a). Therefore, the transitivity of $\dashv$ noted in Proposition 2(T) implies

$$\left[e^x \dashv \ln(x + 1) + 1\right][0]\,.$$

By Proposition 3(c) and Proposition 3(a), $[x + 1 \dashv 1 + \ln(\ln(x + 1) + 1)][0]$. Therefore, by Proposition 2(T),

$$[e^x \dashv 1 + \ln(\ln(x + 1) + 1)][0]\,.$$

Since

$$1 + \ln(\ln(x + 1) + 1) = \ln(e) + \ln(\ln(x + 1) + \ln(e)) = \ln(e\ln(e(x + 1))) = \ln\left(\ln(ex + e)^e\right),$$

it follows that

$$\left[e^x \dashv \ln\left(\ln(ex+e)^e\right)\right][0]\,.$$

By Proposition 3(c), exponentiation of both sides twice yields

$$\left[e^{e^{e^x}} \dashv (ex+e)^e\right][0]\,.$$

This means that $f'(0) = g'(0)$ where $g(x) = (ex+e)^e$. And g is easy to differentiate:

$$g'(x) = e^2(ex+e)^{e-1}\,.$$

Therefore, $f'(0) = g'(0) = e^{e+1}$. □

The next problem is similar but more general.

Problem 5. *Find the derivative of the function* $f(x) = e^{x^2}\sin(x^3)$ *at any point* $x = m$.

Solution. We can solve this problem without directly differentiating f, which explains the "indirect differentiation" part of this article's title. Our approach is to find another function that's tangent to f at $x = m$.

By Problem 1,

$$\left[x^2 \dashv 2mx - m^2\right][m]\,, \tag{1}$$

so Proposition 3(b) implies that

$$\left[3mx^2 \dashv 6m^2x - 3m^3\right][m]\,. \tag{2}$$

Now, note that $[x^3 \dashv 0][0]$, so $\left[(x-m)^3 \dashv 0\right][m]$ by Proposition 4. In other words,

$$[x^2 - 3mx^2 + 3m^2x - m^3 \dashv 0][m]\,.$$

By (2) and Proposition 3(d), $[x^3 + 3m^2x - m^3 \dashv 6m^2x - 3m^3][m]$, and so

$$[x^3 \dashv 3m^2x - 2m^3][m]\,. \tag{3}$$

By (1), (3) and Proposition 3(c),

$$\left[e^{x^2} \dashv e^{2mx-m^2}\right][m] \qquad \text{and} \qquad \left[\sin(x^3) \dashv \sin(3m^2x - 2m^3)\right][m]\,.$$

Hence,

$$\left[e^{x^2}\sin(x^3) \dashv e^{2mx-m^3}\sin(3m^2x - 2m^3)\right][m]\,.$$

The right-hand side is a function of the form $g(x) = e^{ax+b}\sin(cx+d)$ which, as far as differentiation is concerned, is easier to deal with compared to $e^{x^2}\sin(x^3)$. Let the left- and right-side functions be denoted by f and g, respectively; then $f'(m) = g'(m)$. Now,

$$g'(x) = 2me^{2mx-m^2}\sin(3m^2x - 2m^3) + 3m^2e^{2mx-m^2}\cos(3m^2x - 2m^3)\,,$$

Therefore, $f'(m) = g'(m) = me^{m^2}\left(2\sin(m^3) + 3m\cos(m^3)\right)$. □

4 Closing remark

By use of this tangency equivalence relation $\dashv$, it was possible to deduce information about the first-order derivatives of multiple functions and their tangent curves. If one were to extend this definition, then one could create an equivalence class of functions that not only possesses equal values and derivatives at x_0 but also has equal higher-order derivatives. In particular for any non-negative integer n, let

$$f \overset{n}{\dashv} g$$

for real functions f and g that are nth differentiable at some real number x_0 mean that

$$\begin{gathered} f(x_0) = g(x_0) \\ f'(x_0) = g'(x_0) \\ \vdots \\ f^{(n)}(x_0) = g^{(n)}(x_0)\,. \end{gathered}$$

The reader is invited to investigate the properties and applications of these relations.

References

[1] L. Lovász, J. Pelikán, and K. Vesztergombi, *Discrete Mathematics: Elementary and Beyond*, Springer, 2003.