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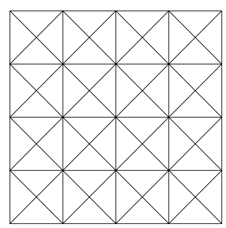
Problems 1751–1760

Q1751 Show that it is possible to partition the set of unit fractions

$$F = \left\{ \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \right\}$$

into finite subsets in such a way that the sum of the fractions in each subset is 1. ("Partition" means that your subsets must, collectively, contain each of the numbers in F exactly once.)

Q1752 The diagram shows a 4×4 grid of squares, each with its two diagonals drawn. If we draw a similar figure starting with a $(2n) \times (2n)$ grid of squares, then how many triangles does the figure contain?



Q1753 Find all positive solutions of the simultaneous equations

$$x + y + z = 10\ , \quad xz = y^2\ , \quad x^2 + y^2 = z^2\ .$$

Q1754 There is a stable of n^2 horses. Show how to rank the n fastest of these by arranging $2n$ races, each race to involve at most n horses.

Assume that there is always just one winner; and that horses always run "true to form"; that is, if horse A *ever* beats horse B, then horse A *always* beats horse B.

Q1755 The circumference of a circle is divided into four arcs whose midpoints are A, B, C, D in that order around the circle. Prove that AC is perpendicular to BD.

Q1756 Jack and Jill are going to climb a hill. Two hills, actually. Jill says to Jack, "We'll climb from sea level to the top of Intermediate Hill, 151 metres – uphill all the way! Then we'll descend 97 (vertical) metres into a saddle; then climb uphill again to the Goat House Cave which is 444 metres above sea level." "Sounds fine," replies Jack. "I can manage those two ascents. Of course we don't need to count the descent, that will be easy." A bit later, Jill continues, "Actually, I made a mistake – Intermediate Hill is 251 metres above sea level. However, I'm quite certain I got the other two figures correct." "Now I'm worried," says Jack. "I don't think I can handle the extra 100 metres of climbing."

Comments please! *But without calculation.*

Q1757 Let a, b, c be non–zero real numbers.

(a) Prove that if the quadratics ax^2+bx+c and bx^2+cx+a have a real root in common, then it is also a root of cx^2+ax+b.

(b) Find the monic cubic whose three roots are the *other* (non–shared) roots of the three quadratics in (a). "Monic" means that the coefficient of x^3 must be 1.

Q1758 For $x = 2\pi/13$, evaluate

$$(\sec 2x + \sec 3x)(\sec 4x + \sec 6x)(\sec x + \sec 5x)\ .$$

Q1759 A bag contains 16 balls, 4 each of 4 different colours. Four balls are chosen at random – this counts as one move. If they are all of the same colour, then they are discarded; if not, then all four are returned to the bag. The same kind of move is performed repeatedly. How many moves, on average, does it take until the bag is empty?

(This problem was inspired by the "Connections" puzzle which appears daily in the *New York Times*.)

Q1760 Find, if possible, a sequence $a_1, a_2, a_3, \ldots, a_n$ of n positive real numbers satisfying the equations

$$\begin{aligned} a_1 + a_2 + a_3 + \cdots + a_n &= 2024 \\ a_1 + 2a_2 + 3a_3 + \cdots + na_n &= 2025 \\ a_1 + 4a_2 + 9a_3 + \cdots + n^2a_n &= 2026\ . \end{aligned}$$