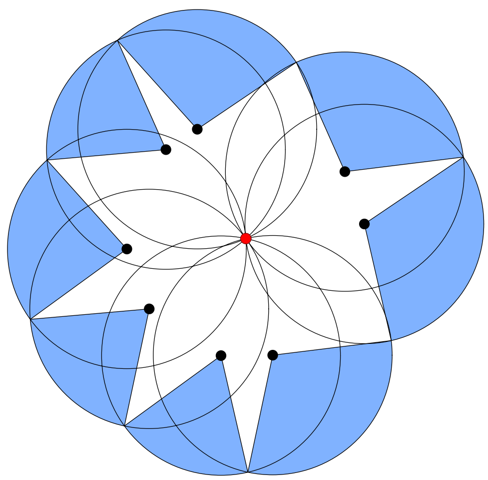
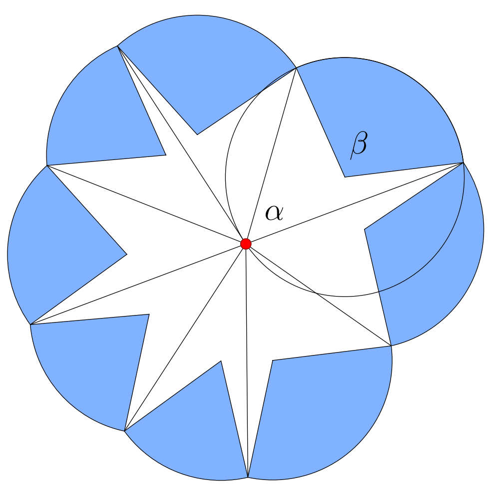
Parabola Volume 60, Issue 3 (2024)

Solutions 1741–1750

Q1741 There are n circles of radius 1, all intersecting at a point P: the diagram gives an example with $n = 8$, where P is the red dot in the middle. Lines are drawn from the centre of each circle to its points of intersection (other than P) with the two neighbouring circles. The coloured region lies between these lines and the "external" arcs of the circles. Supposing that none of these "external" arcs contains P, what is the area of the coloured region?



SOLUTION In the following diagram, for the sake of clarity, we have eliminated the "internal" arcs of all the circles, except for one.



The area of each sector is $\pi r^2(\beta/2\pi)$, where β is the angle subtended by the corresponding arc at its centre. By the angle–at–centre–angle–at–circumference theorem, the angle α subtended by each arc at P is equal to $\beta/2$; and the radius in this case is $r = 1$; so the area of one sector is α. Therefore, the total area of the coloured region is equal to the sum of all the angles at P, which is 2π.

Q1742 Show that if the product of two, three or four consecutive positive integers is increased by 1, then the result is (respectively) never, sometimes or always a square.

SOLUTION For the first question, we note that

$$n^2 < n(n+1)+1 < (n+1)^2 \; ;$$

so $n(n+1)+1$ lies between two consecutive squares, and cannot itself be a square.

The second is really just a matter of trial and (no) error: we easily find, for example, that $1\times2\times3+1$ is not a square but $2\times3\times4+1$ is.

For the third question, we need to consider a product like $(n-1)n(n+1)(n+2)$. We can often greatly simplify mathematical problems by making use of any available symmetry. In this case, the product does not look very symmetric (there is one term below n and two above); but it is much more so if we substitute $m = n+\frac{1}{2}$. Then we have

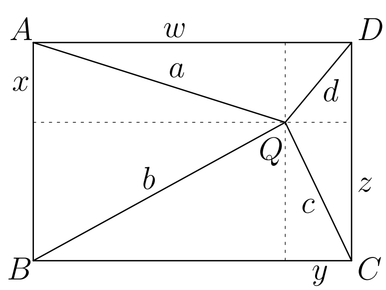
$$\begin{aligned}(n-1)n(n+1)(n+2)+1 &= \Big(m-\frac{3}{2}\Big)\Big(m-\frac{1}{2}\Big)\Big(m+\frac{1}{2}\Big)\Big(m+\frac{3}{2}\Big)+1\\ &= \Big(m^2-\frac{1}{4}\Big)\Big(m^2-\frac{9}{4}\Big)+1\\ &= m^4-\frac{5}{2}m^2+\frac{25}{4}\\ &= \Big(m^2-\frac{5}{4}\Big)^2\\ &= (n^2+n-1)^2 \, ,\end{aligned}$$

which is the square of an integer.

Solution received from Titu Zvonaru, Comăneşti, Romania.

Q1743 A pyramid has a rectangular base $ABCD$ and a vertex P (which does not have to be directly above the centre of the base). If AP, BP, CP have lengths $63, 60, 16$, then find the length of DP.

SOLUTION Let Q be the foot of the perpendicular from P to the plane of $ABCD$, and label lengths as in the diagram; also, let q be the length of PQ.



Using Pythagoras' Theorem many times, we have

$$a^2+c^2 = (w^2+x^2)+(y^2+z^2) = (w^2+z^2)+(x^2+y^2) = b^2+d^2$$

and so

$$|AP|^2 + |CP|^2 = (a^2 + q^2) + (c^2 + q^2) = (b^2 + q^2) + (d^2 + q^2) = |BP|^2 + |DP|^2 \ .$$

For the figures given in the question, $|DP|^2 = 63^2 + 16^2 - 60^2 = 625$ and so $|DP| = 25$.

Solution received from Titu Zvonaru, Comăneşti, Romania.

Comment. The relation $a^2+c^2 = b^2+d^2$ is still true if the point Q lies outside $ABCD$. We invite readers to check this for themselves.

Q1744 Prove that any multiple of 36 can be written as the sum of four different perfect cubes. Note that perfect cubes don't have to be positive: they can be negative, or even zero. For example, $-8 = (-2)^3$ and $0 = 0^3$ are allowable summands.

SOLUTION Consider $36k$, a multiple of 36. Observe that if a is an integer and we expand $(2k + a)^3 + (2k - a)^3$, all the k^2 and constant terms drop out:

$$(2k + a)^3 + (2k - a)^3 = 16k^3 + 12ka^2 \ .$$

Now do the same thing starting with an integer b instead of a, and subtract the two expressions. The k^3 terms also drop out and we have

$$(2k + a)^3 + (2k - a)^3 - (2k + b)^3 - (2k - b)^3 = 12k(a^2 - b^2) \ ;$$

this will equal our "target" if we choose $a = 2$ and $b = 1$. Since cubes can be negative, the subtractions can be rewritten as additions,

$$36k = (2k + 2)^3 + (2k - 2)^3 + (-2k + 1)^3 + (-2k - 1)^3 \ ,$$

meeting the requirements of the question. To complete the problem, we need to confirm that the four cubes we have written down are all different. It is obvious that the first two are unequal, as are the last two. Moreover, the first two (which are even) cannot be equal to the last two (which are odd). So all four cubes are different, and we are finished.

Q1745 We are given the set of all fractions with numerator 1 and prime denominators:

$$S = \left\{ \frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{11}, \ldots \right\} .$$

You are allowed to take the difference of any two numbers in S and include it in S; and then to repeat the process, using your newly produced fraction if you wish. For example, you could obtain

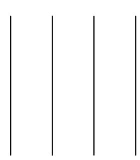
$$\frac{1}{2} - \frac{1}{7} = \frac{5}{14} \quad \text{and then} \quad \frac{5}{14} - \frac{1}{7} = \frac{3}{14} \quad \text{and also} \quad \frac{1}{3} - \frac{1}{5} = \frac{2}{15}$$

and so on. Is it possible to obtain the fraction $\dfrac{1}{2024}$?

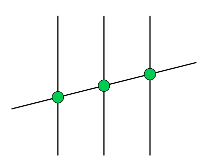
SOLUTION Write all fractions in lowest terms. When two fractions are subtracted, the denominator of the difference will be the least common multiple of the two denominators – or perhaps a factor of this least common multiple, as it is possible to have some cancellation between the numerator and denominator of the difference. Among the initial fractions, no denominator is ever divisible by the square, or any higher power, of a prime number; so the same will be true after taking the difference of two fractions; and after taking another difference; and so on. Therefore, it will never be possible to obtain a fraction with a factor 2^3 in its denominator, as $\frac{1}{2024}$ does.

Q1746 There are four lines and 2024 red points in a plane. None of the red points lies on any of the four lines; but the lines divide the plane into various regions, each containing an equal number of the red points. Any point where two or more of the lines intersect is coloured green. How many green points are there?

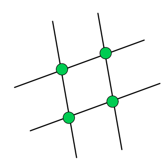
SOLUTION A bit of trial and error shows that there are six essentially different configurations of four lines dividing the plane into various numbers of regions. They are shown in the following diagram, each configuration marked with the number of regions it creates.



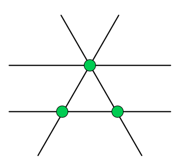
[5 regions]



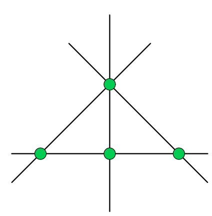
[8 regions]



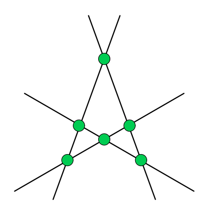
[9 regions]



[9 regions]



[10 regions]



[11 regions]

However in the present problem, the first configuration is impossible, as $2024 = 2^3 \times 11 \times 23$ red points cannot be partitioned into 5 sets of equal size. For similar reasons, the only possibilities are those which divide the plane into 8 or 11 regions. Therefore, there are either 3 or 6 green points.

Q1747 Write a_k for the last digit of the sum of the digits of k. This gives a sequence

$$a_1, a_2, \ldots = 1, 2, 3, 4, 5, 6, 7, 8, 9, 1, 2, 3, 4, 5, 6, 7, 8, 9, 0, 2, 3, 4, \ldots .$$

Show that this sequence never contains the same digit more than twice consecutively.

SOLUTION For any positive integer k, we denote by $s(k)$ the sum of the digits of k. Suppose that k ends with exactly m nines. (So either the previous digit is something other than nine, or there is no previous digit.) When we add 1 to k, all these nines will be replaced by zeros, and the previous digit will be increased by 1. (Or a 1 will appear where there was no previous digit.) Therefore,

$$s(k+1) - s(k) = 1 - 9m \, .$$

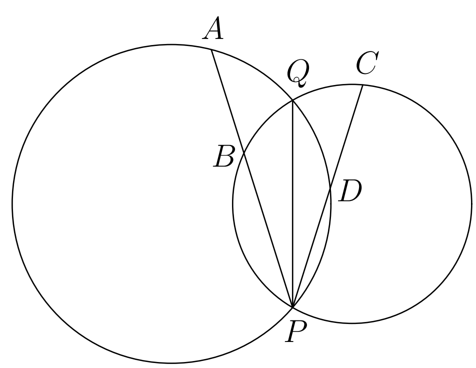
The only way in which we can have $a_k = a_{k+1}$ is if the difference between $s(k+1)$ and $s(k)$ is a multiple of 10, that is, $m = 9, 19, 29, \ldots$, meaning that k ends in this many nines. But in this case $k+1$ does not end in any nines at all, it ends in a zero: so it is impossible to have $a_k = a_{k+1}$ and also $a_{k+1} = a_{k+2}$. That is, the sequence never contains the same digit more than twice consecutively.

Comment. This also shows that the first time we have the same digit twice consecutively in the sequence is when $k = 999999999$: then $s(k) = 81$, $a_k = 1$ and $s(k+1) = s(1000000000) = 1$, $a_{k+1} = 1$.

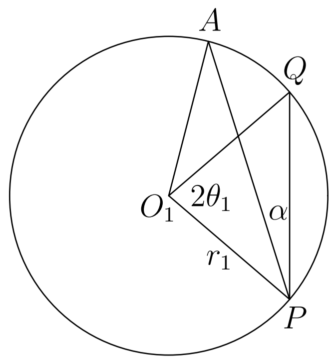
Q1748 Three cyclists Andy, Bobby and Cassie ride around a circular track, all in the same direction, at respective speeds of $24, 40, 50$ kilometes per hour. At certain times all three of them are together. In between two successive triple meetings, how many times are there when two of the cyclists meet?

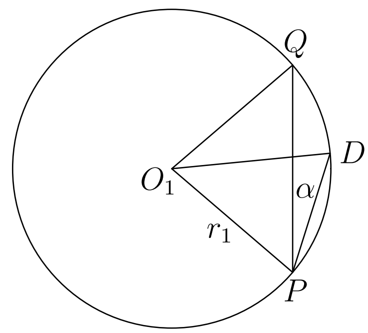
SOLUTION Let t be the time taken for Bobby to catch up with Andy (that is, to get ahead of him by one whole lap of the track). As Bobby travels 16 km/h faster than Andy while Cassie is 26 km/h faster, Cassie gets $\frac{26}{16} = \frac{13}{8}$ laps ahead of Andy in this time. For a triple meeting to occur, both Bobby and Cassie must be a whole number of laps ahead of Andy. The first occasion on which this will happen is at time $8t$, when Bobby will have advanced on Andy by 8 laps and Cassie will have advanced on Andy by 13 laps. Since we don't count either the initial or final triple meeting, this means that Bobby will have met Andy 7 times and Cassie will have met him 12 times. Moreover, Cassie will have advanced on Bobby by 5 laps and will therefore have met her 4 times. So the total number of meetings between two of the three cyclists will have been $7 + 12 + 4 = 23$.

Q1749 In the diagram, two circles intersect at P and Q; lines through P intersect one of the circles at A and D, and the other at B and C, as shown. If $\angle APQ = \angle CPQ$, then prove that the line segments AB and CD are of equal length.



SOLUTION We extract part of the diagram and write $\angle APQ = \alpha$; also, r_1 for the radius of the circle and $2\theta_1$ for the angle subtended at the centre by PQ; see the left–hand diagram below.





Then we have $|PQ| = 2r_1 \sin\theta_1$; moreover, looking at angles on the arc AQ shows that $\angle AO_1Q = 2\alpha$ and hence

$$|AP| = 2r_1 \sin(\theta_1 + \alpha)\ .$$

Looking at the right–hand diagram above and using similar ideas,

$$|DP| = 2r_1 \sin(\theta_1 - \alpha)\ .$$

Exactly the same arguments hold for the other circle: writing r_2 for its radius and $2\theta_2$ for the angle subtended by $|PQ|$, we have

$$|PQ| = 2r_2 \sin\theta_2\ , \quad |CP| = 2r_2 \sin(\theta_2 + \alpha)\ , \quad |BP| = 2r_2 \sin(\theta_2 - \alpha)\ .$$

Therefore, $2r_1 \sin\theta_1 = 2r_2 \sin\theta_2$; and using formulae for the sine of a sum and difference, we have

$$\begin{aligned} |AB| &= |AP| - |BP| \\ &= (2r_1 \sin\theta_1 \cos\alpha + 2r_1 \cos\theta_1 \sin\alpha) - (2r_2 \sin\theta_2 \cos\alpha - 2r_2 \cos\theta_2 \sin\alpha) \\ &= (2r_2 \sin\theta_2 \cos\alpha + 2r_2 \cos\theta_2 \sin\alpha) - (2r_1 \sin\theta_1 \cos\alpha - 2r_1 \cos\theta_1 \sin\alpha) \\ &= |CP| - |DP| \\ &= |CD|\ , \end{aligned}$$

as required.

Alternative solution by Titu Zvonaru, Comăneşti, Romania. Referring to the same diagram, note that in the left–hand circle, angles $\angle APQ$ and $\angle DPQ$ are equal, so the lengths $|AQ|$ and $|DQ|$ are equal. Considering the right–hand circle in the same way, $|BQ|$ and $|CQ|$ are equal. Since quadrilateral $PDQA$ is cyclic, we have

$$\angle PAQ = 180^\circ - \angle PDQ \quad \text{and hence} \quad \angle BAQ = \angle CDQ\ ;$$

for similar reasons, $\angle ABQ = \angle DCQ$. Therefore, triangles $\triangle ABQ$ and $\triangle DCQ$ are congruent, and we have $|AB| = |CD|$ as required.

Q1750 For each letter of the alphabet $A, B, C, \ldots, Z$, we refer to the previous and subsequent letters in alphabetical order as its "neighbours". For example, the neighbours of D are C and E; Z has only one neighbour, Y. We wish to order all 26 letters in such a way that each letter in our list, except the first, is preceded at some stage by at least one of its neighbours. For instance,

$$DCBEAFG \cdots XYZ$$

is an allowable list, but

$$BADECFG \cdots XYZ$$

is not, because D is preceded neither by C nor by E. In how many ways can the alphabet be ordered in this fashion?

SOLUTION If the first letter of our sequence is D, then this must be followed (not necessarily straight away) by C, B, A *in that order*, and also by $E, F, G, \ldots, X, Y, Z$ *in that order*. The only choice we have to make is which 3 of the 25 places following the D are to be occupied by C, B, A, and there are $C(25, 3)$ ways to make this choice. In the same way, if we start with the $(k + 1)$th letter of the alphabet, then there are $C(25, k)$ possible sequences. Therefore, the total number of sequences is

$$C(25, 0) + C(25, 1) + C(25, 2) + \cdots + C(25, 5) \, .$$

This is the sum of the entries in row 25 of Pascal's triangle; it is fairly well known that the sum is 2^{25}, and this is the answer to our problem.

Alternative solution. At any stage in the process of constructing our sequence, we must write a letter which is a neighbour of at least one letter used so far; therefore, the letters used up to any stage must form a block of adjacent letters from the alphabet. For example, the reason why the second sequence in the question is not permissible is that, after writing the third letter, we have used A, B, D, which is not a block of consecutive letters. It follows that the first 25 letters of our sequence must consist of 25 adjacent letters from the alphabet; which means that the 26th and last letter can only be A or Z.

Now a sequence ending in Z is permissible if and only if the previous entries form a permissible sequence on the 25–letter alphabet $A, B, \ldots, X, Y$; and a sequence ending in A is permissible if and only if the previous entries form a permissible sequence on the 25–letter alphabet $B, C, \ldots, Y, Z$. Therefore, the number of allowable sequences on a 26–letter alphabet is twice the number on a 25–letter alphabet; which by the same argument is twice the number on a 24–letter alphabet; and so on. Since with a single letter A we have one possible sequence, the total number for 26 letters is 2^{25}, as we found earlier.