Parabola Volume 61, Issue 1 (2025)

sin(18°): Simplest proof and the pentagon conundrum

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1 Introduction

When addressing the general question of how to inscribe regular polygons in a circle, we inevitably face the intriguing and non-intuitive case of the pentagon. Typically, without much explanation, our teachers instructed us to follow the geometric construction illustrated in the sequence of three figures below.

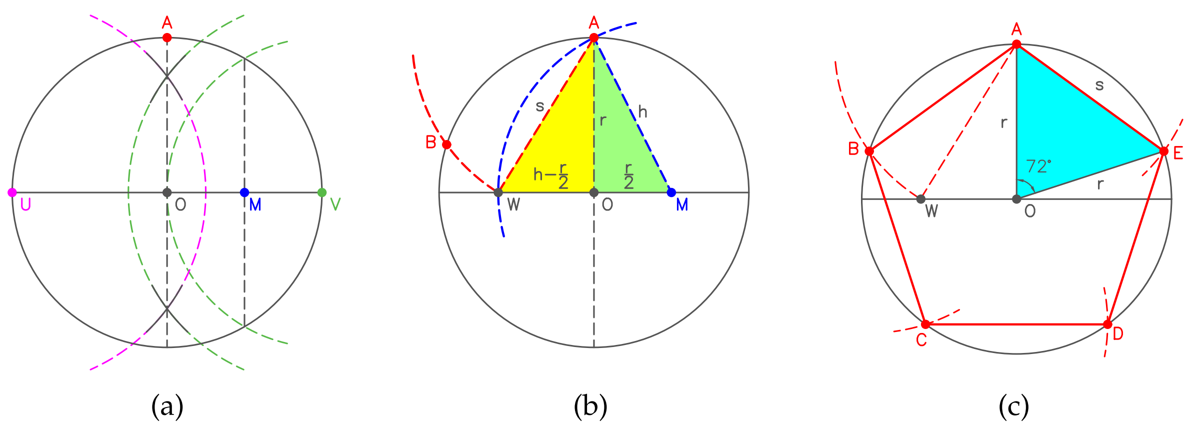

Figure 1: Pentagon construction method

For the inquisitive mind, however, this geometric process seems to come out of nowhere. How on earth could those few simple compass moves split the circumference exactly into five parts? That haunted me[4] for a very long time until one day, or rather one night, when a flash of inspiration crossed my mind, and I devised the proof that we are about to present.

The cornerstone to the demonstration is proving that $\cos(72°)$, or its complement $\sin(18°)$, is exactly $\frac{\sqrt{5}-1}{4}$. In order to understand this deep connection, first consider the

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right triangles OAM and OAW in Figure 1(b). By employing Pythagoras' Theorem to both of them, the following result can be derived:

$$s^2 = \frac{r^2}{2}\left(5 - \sqrt{5}\right)$$

which, according to the construction method, represents the square of the side of the pentagon in terms of its circumradius. Clearly, a second relationship for s^2 can be derived by assuming a perfectly constructed pentagon. To achieve this, examine triangle OAE in Figure 1(c) and apply the Cosine Rule:

$$s^2 = 2r^2\big(1 - \cos(72°)\big)\,.$$

Now, by equating these two expressions for s^2, it can be concluded that the pentagon construction method is exact if and only if

$$\cos(72°) = \sin(18°) = \frac{\sqrt{5}-1}{4}$$

and that's where I got stuck... That annoying pebble stayed in my shoe for ages, occasionally resurfacing in my mind and prompting me to try different solution paths. Eventually, I found a (very) simple way to solve this conundrum. Moreover, my students (and co-authors) Emilly and Renan helped me realise that the proof was fairly unique and insightful enough to share with *Parabola* readers. We hope you enjoy it.

2 sin(18°) - short and sweet

The proof begins by acknowledging the obvious fact that the cosine function nullifies whenever its argument is an odd multiple of 90°:

$$\cos(1 \times 90°) = \cos(5 \times 18°) = 0$$

$$\cos(3 \times 90°) = \cos(5 \times 54°) = 0$$

$$\cos(5 \times 90°) = 0$$

$$\cos(7 \times 90°) = \cos(5 \times 126°) = 0$$

$$\cos(9 \times 90°) = \cos(5 \times 162°) = 0$$

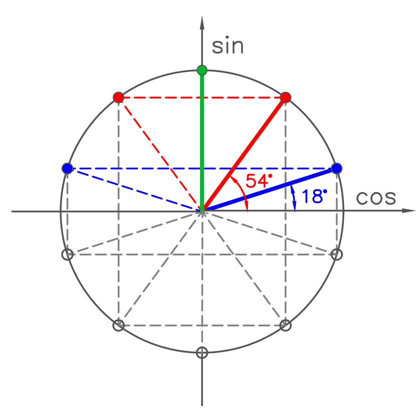

Figure 2: Solutions for $\cos(5\theta) = 0$

Further increases in the argument, whether positive or negative, would result in five-fold angles whose cosines yield values symmetrical to those already highlighted in Figure 2. This assertion, which simply reflects the even property of cosines, implies that the well-known five-fold angle cosine equation (see Appendix):

$$\cos(5\theta) = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta = \cos\theta\left(16\left(\cos^2\theta\right)^2 - 20(\cos^2\theta) + 5\right) = 0$$

can provide two distinct, non-trivial, positive exact solutions, corresponding to the first-quadrant angles $\theta = 18°$ and $\theta = 54°$. These two solutions can be easily obtained by first using Bhaskara's Formula to solve for $\cos^2\theta$:

$$\cos^2\theta = \frac{5 \pm \sqrt{5}}{8}$$

and then applying the Pythagorean trigonometric identity:

$$\cos^2\theta + \sin^2\theta = 1$$

to get the desired first-quadrant elegant sine solutions:

$$\sin^2\theta = 1 - \cos^2\theta = \frac{3 \pm \sqrt{5}}{8} = \left(\frac{\sqrt{5} \pm 1}{4}\right)^2 \Rightarrow \begin{cases} \sin(54°) = \dfrac{\sqrt{5}+1}{4} \\ \boxed{\sin(18°) = \dfrac{\sqrt{5}-1}{4}} \end{cases}$$

This final result is my long-sought exact solution for the $\sin(18°)$ or, equivalently, its complement $\cos(72°)$. It is also, of course, my ticket to peace, as it unequivocally proves that the Euclidean millennia-lasting geometric construction of the regular pentagon is indeed exact – a concept I long struggled to embrace.

3 Measuring up against popular proofs

A thorough review of specialized literature and more general internet sources revealed a plethora of proofs for the exact value of $\sin(18°)$. Nonetheless, every single one of those we found can be distilled into just three fundamental hybrid methods: one geometric-algebraic and two trigonometric-algebraic.

The geometric-algebraic approach involves the construction of specific geometric figures, along with the application of some basic maths principles, such as similarity of triangles, the Pythagora's theorem and Bhaskara's Formula [1, 2]. While the beauty of this method should be recognized, it requires intricate constructions and a good deal of mental interpretation.

The second and most popular of the three starts with a distinctive cofunction identity, $\sin(2\theta) = \cos(3\theta)$, which can be confusing to beginners, given that it is valid only when $\theta = 18°$ [3, 4]. This identity allows for substantial simplification in the ensuing manipulations with double and triple angle identities, reducing them to a solvable quadratic equation. While the method is interesting, it is neither shorter nor particularly intuitive, requiring significant trigonometric skill to devise.

The third and most intricate of the three methods uses $\sin(5\theta) = 1$, leading to a quintic polynomial equation. Although this approach is effective, it requires careful reasoning to recognize the perfect square of a quadratic polynomial within the equation [5, 6]. In contrast, here we entirely bypass these more complex mathematical operations by opting for the cosine function instead of the sine. This choice also results in a quintic equation, but a simpler one, which significantly eases subsequent development. Thus, our approach stands out as the simplest, eliminating the need for geometric visualization, minimizing operations, and avoiding any 'magical' trick.

4 Pedagogical perk

The proof for $\sin(18°)$, as presented here, in connection with the construction of the pentagon, emphasizes the use of several fundamental concepts from school mathematics, including Bhaskara's Formula, the cosine rule, Pythagoras' Theorem, perfect square polynomials, the Pythagorean trigonometric identity, cofunction identities, multiple-angle trigonometric identities, even/odd functions, and, of course, the geometric construction of the pentagon itself. This approach, carried out without any complex mathematical manipulation, allows teachers to challenge students while reviewing a significant portion of elementary mathematics.

Acknowledgements

We would like to express our sincere gratitude to Tom Edgar for his warm and invaluable advice in guiding us towards the *Parabola* journal.

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Appendix

$$
\begin{aligned}
1 &= \sin^2(a) + \cos^2(a) \\
\cos(a+b) &= \cos(a)\cos(b) - \sin(a)\sin(b) \\
\sin(a+b) &= \cos(a)\sin(b) + \cos(b)\sin(a)
\end{aligned}
$$

$$
\begin{aligned}
\sin^2\theta &= 1 - \cos^2\theta \\
\sin(2\theta) &= 2\cos\theta\sin\theta \\
\cos(2\theta) &= 2\cos^2\theta - 1
\end{aligned}
$$

$$
\begin{aligned}
\sin(3\theta) &= \sin(\theta + 2\theta) \\
&= \cos\theta(2\cos\theta\sin\theta) + (2\cos^2\theta - 1)\sin\theta \\
&= (4\cos^2\theta - 1)\sin\theta
\end{aligned}
$$

$$
\begin{aligned}
\cos(3\theta) &= \cos(2\theta + \theta) \\
&= (2\cos^2\theta - 1)\cos\theta - (2\cos\theta\sin\theta)\sin\theta \\
&= 4\cos^3\theta - 3\cos\theta
\end{aligned}
$$

$$
\begin{aligned}
\cos(5\theta) &= \cos(2\theta + 3\theta) \\
&= (2\cos^2\theta - 1)(4\cos^3\theta - 3\cos\theta) - (2\cos\theta\sin\theta)(4\cos^2\theta - 1)\sin\theta \\
&= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta
\end{aligned}
$$