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Means in mathematics and relationships between them

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1 Introduction

In mathematics, a mean is a quantity which represents the average value of a collection of two or more real numbers. The value of a mean is a real number which lies between the smallest and largest elements of the given collection.

There are many types of mean. The most commonly used and well-known means are the arithmetic mean, the geometric mean, the harmonic mean, and the quadratic mean.

The point of this article is not only to recall how these means are defined but also to present and elegantly prove the relationships between them.

Throughout this article, the length of the line segment XY will be denoted as $|XY|$.

2 The arithmetic mean A

The arithmetic mean $A(x_1, x_2, \ldots, x_n)$ of a finite collection of real numbers $x_1, x_2, \ldots, x_n$ is defined as the sum of numbers in the collection divided by the count n of numbers in the collection. This can be symbolically expressed in the following way:

$$A(x_1, x_2, \ldots, x_n) = \frac{x_1 + x_2 + \cdots + x_n}{n} = \frac{1}{n}\sum_{i=1}^{n} x_i.$$

We will write briefly A instead of $A(x_1, x_2, \ldots, x_n)$ in the figures, and similarly for the other means.

Consider two positive real numbers x and y. The arithmetic mean

$$A(x, y) = \frac{x+y}{2}$$

can be easily interpreted graphically as follows. Let x and y be the length of the line segments KN and NL, respectively. Arrange these line segments so that they form the line segment KL of length $x + y$. Let M be the midpoint of KL and construct the semicircle k with diameter KL.

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Let O be the intersection of k with the perpendicular drawn from the point M to the line segment KL. The radius $|MK| = |MO|$ of k is clearly equal to $\frac{1}{2}(x+y) = A(x,y)$. See Figure 1 for illustration.

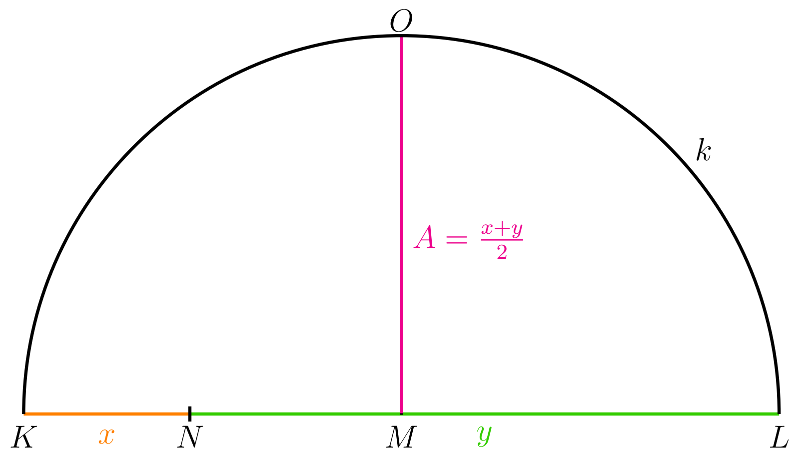

Figure 1: Geometric representation of the arithmetic mean of x and y

3 The geometric mean G

The geometric mean $G(x_1, x_2, \ldots, x_n)$ of positive real numbers $x_1, x_2, \ldots, x_n$ is the n-th root of the product of these n numbers:

$$G(x_1, x_2, \ldots, x_n) = \sqrt[n]{x_1 x_2 \cdots x_n} = \left(\prod_{i=1}^{n} x_i \right)^{\frac{1}{n}}.$$

We can also graphically illustrate the geometric mean $G(x,y) = \sqrt{xy}$ of two positive real numbers x and y in the following way.

Using Figure 1 as the set-up, draw a line segment perpendicular to KL that passes through the point N and let P be its intersection with the semicircle k. Since the triangles KNP and PNL are similar, it follows that

$$\frac{|KN|}{|NP|} = \frac{|PN|}{|NL|}.$$

Hence, $|NP|^2 = |KN||NL|$ and consequently

$$|NP| = \sqrt{xy} = G(x,y).$$

This is essentially the *right triangle altitude theorem* applied to the triangle KLP.

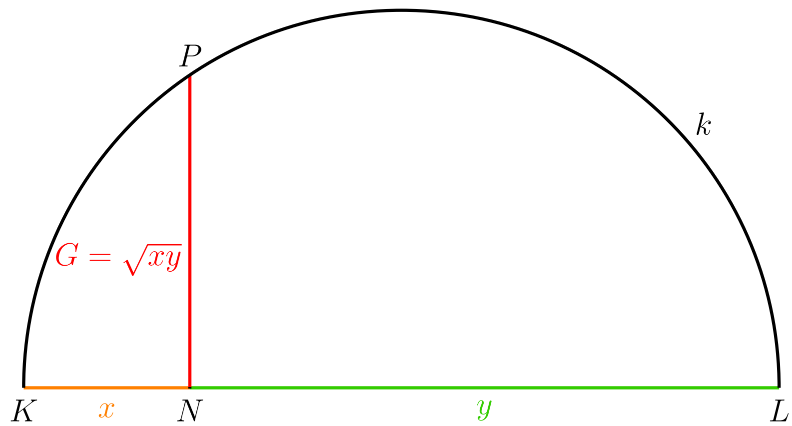

Figure 2: Geometric representation of the geometric mean of x and y

4 The harmonic mean H

The harmonic mean $H(x_1, x_2, \ldots, x_n)$ of positive real numbers $x_1, x_2, \ldots, x_n$ is defined as the reciprocal of the arithmetic mean of the reciprocals of $x_1, x_2, \ldots, x_n$. This can be expressed as

$$H(x_1, x_2, \ldots, x_n) = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}} = \frac{n}{\displaystyle\sum_{i=1}^{n} \frac{1}{x_i}} .$$

The harmonic mean of two positive numbers x and y is

$$H(x, y) = \frac{2}{\frac{1}{x} + \frac{1}{y}} = \frac{2xy}{x + y}$$

and again a geometric interpretation using the same semicircle k is possible.

Let K, L, M, N, P, O denote the same points as in Figures 1 and 2. Consider the right triangle NMP and let R be the foot of the perpendicular drawn from the point N to its hypotenuse MP. See Figure 3 for illustration.

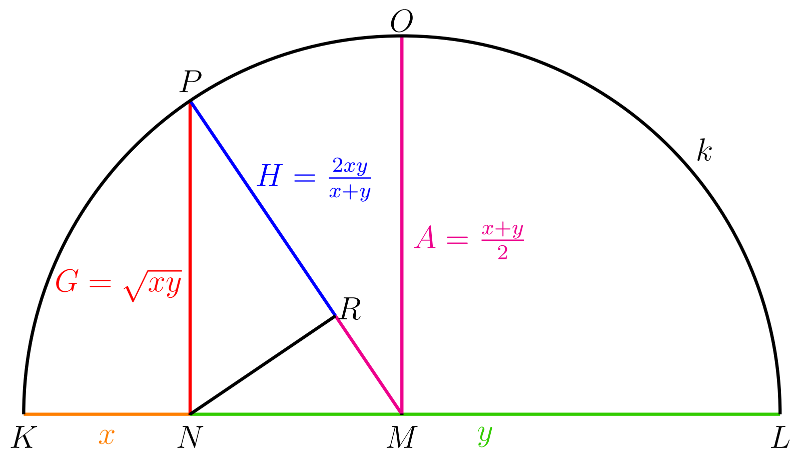

Figure 3: Geometric representation of the harmonic mean of x and y

Obviously, $|MP| = |MO| = \frac{1}{2}(x+y)$. By using the Pythagorean theorem for the right triangle RPN and then applying the right triangle altitude theorem to the triangle NMP, we obtain

$$\begin{aligned} |PN|^2 &= |PR|^2 + |RN|^2 \\ &= |PR|^2 + |RP|\,|RM| \\ &= |PR|\left(|PR| + |RM|\right). \end{aligned}$$

It follows that

$$xy = |PN|^2 = |PR|\,|PM| = \frac{1}{2}(x+y)|PR|$$

and consequently

$$|PR| = \frac{2xy}{x+y} = H(x,y)\,,$$

giving the harmonic mean $H(x,y)$ of the numbers x and y an interpretation as the length of the line segment PR.

5 The quadratic mean Q

The quadratic mean $Q(x_1, x_2, \ldots, x_n)$ of a finite collection of real numbers $x_1, x_2, \ldots, x_n$ is the square root of the arithmetic mean of the squares of the given numbers:

$$Q(x_1, x_2, \ldots, x_n) = \sqrt{\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n}} = \left(\frac{1}{n}\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}.$$

The quadratic mean of two positive real numbers x and y is

$$Q(x,y) = \sqrt{\frac{x^2+y^2}{2}}\,.$$

This value can also be represented as the length of a certain line segment constructed through the same semicircle k.

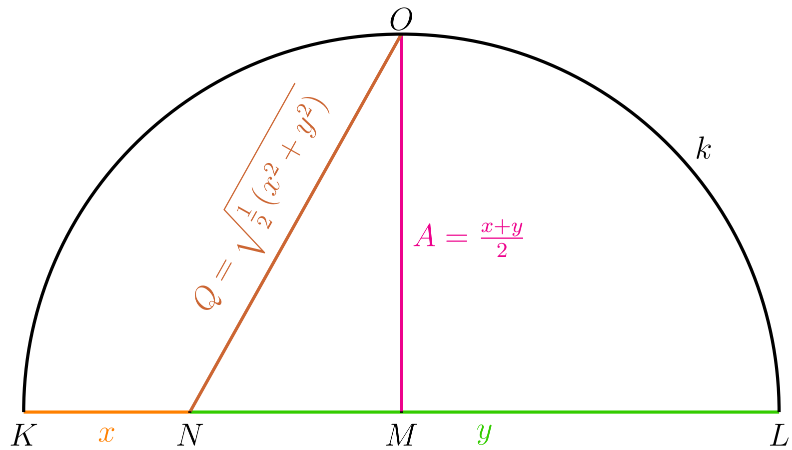

Figure 4: Geometric representation of the quadratic mean of x and y

If K, L, M, N, P, O denote the same points as in Figure 3, the triangle NMO is right-angled. Assuming $x \le y$, the lengths of its legs are

$$|NM| = |KM| - |KN| = \tfrac{1}{2}(x+y) - x = \tfrac{1}{2}(y-x)\,, \quad |MO| = \tfrac{1}{2}(x+y)\,.$$

By the Pythagorean Theorem,

$$|NO| = \sqrt{|NM|^2 + |MO|^2} = \sqrt{\left(\frac{1}{2}(y-x)\right)^2 + \left(\frac{1}{2}(x+y)\right)^2} = \sqrt{\frac{1}{2}(x^2+y^2)} = Q(x,y)\,.$$

6 Some examples

Let us calculate all four means for several concrete collections of numbers. We will start with a collection of two numbers 2 and 17:

$$A(2,17) = \frac{1}{2}(2+17) = 9.5\,, \qquad H(2,17) = \frac{2}{\frac{1}{2}+\frac{1}{17}} \approx 3.579\,,$$

$$G(2,17) = \sqrt{2\cdot 17} \approx 5.831\,, \qquad Q(2,17) = \sqrt{\frac{1}{2}(2^2+17^2)} \approx 12.104\,.$$

By similar computation for the numbers $4, 5, 7, 8$, we obtain

$$A(4,5,7,8) = \frac{1}{4}(4+5+7+8) = 6\,, \quad H(4,5,7,8) = \frac{4}{\frac{1}{4}+\frac{1}{5}+\frac{1}{7}+\frac{1}{8}} \approx 5.572\,,$$

$$G(4,5,7,8) = \sqrt[4]{4\cdot 5\cdot 7\cdot 8} \approx 5.785\,, \quad Q(4,5,7,8) = \sqrt{\frac{1}{4}(4^2+5^2+7^2+8^2)} \approx 6.205\,.$$

Let us extend the previous set of values by adding the number 500 and compute the new means:

$$A(4,5,7,8,500) = \frac{1}{5}(4+5+7+8+500) = 104.8\,,$$

$$G(4,5,7,8,500) = \sqrt[5]{4\cdot 5\cdot 7\cdot 8\cdot 500} \approx 14.114\,,$$

$$H(4,5,7,8,500) = \frac{5}{\frac{1}{4}+\frac{1}{5}+\frac{1}{7}+\frac{1}{8}+\frac{1}{500}} \approx 6.946\,,$$

$$Q(4,5,7,8,500) = \sqrt{\frac{1}{5}(4^2+5^2+7^2+8^2+500^2)} \approx 223.676\,.$$

From the last two examples, we observe that if the values in a collection do not differ greatly, the values of all four means do not differ much either. However, if an extreme number appears in a collection, values of the means get significantly distorted.

In some countries such as the Czech Republic, students' ongoing work is graded using the 5-point scale, where 1 represents the best grade (excellent), and 5 represents the lowest grade (insufficient). After a certain period of study, a final grade is given to each student for each subject. It is often calculated using the (weighted) arithmetic mean, not, for example, the geometric mean. However, we often do not know why this is the case. One may ask whether it is due to tradition, the simplicity of the calculation, or another reason.

If a student received the grades $1,1,1,1,2,4,5$ during their school term for a certain subject, using the four means, the student's final grade is

$$A(1,1,1,1,2,4,5) = \frac{1+1+1+1+2+4+5}{7} \approx 2.143\,,$$

$$G(1,1,1,1,2,4,5) = \sqrt[7]{1\cdot 1\cdot 1\cdot 1\cdot 2\cdot 4\cdot 5} \approx 1.694\,,$$

$$H(1,1,1,1,2,4,5) = \frac{7}{\frac{1}{1}+\frac{1}{1}+\frac{1}{1}+\frac{1}{1}+\frac{1}{2}+\frac{1}{4}+\frac{1}{5}} \approx 1.414\,,$$

$$Q(1,1,1,1,2,4,5) = \sqrt{\frac{1^2+1^2+1^2+1^2+2^2+4^2+5^2}{7}} \approx 2.646\,.$$

By rounding to the nearest integer, the student would receive the grade of 1 using the harmonic mean, the grade of 2 using both the arithmetic and geometric means, and the grade of 3 using the quadratic mean.

7 Inequalities between means

The examples presented above suggest the following relationships between the four means [1]. For positive real numbers $x_1, x_2, \ldots, x_n$,

$$\begin{aligned}\min\{x_1, x_2, \ldots, x_n\} &\le H(x_1, x_2, \ldots, x_n) \le G(x_1, x_2, \ldots, x_n)\\ &\le A(x_1, x_2, \ldots, x_n) \le Q(x_1, x_2, \ldots, x_n) \le \max\{x_1, x_2, \ldots, x_n\}\,.\end{aligned}$$

It can be proved that these relationships are indeed valid and that the equality is achieved if and only if $x_1 = x_2 = \cdots = x_n$.

There are many proofs of this fact. We limit ourselves to the case of two positive numbers x and y. Without loss of generality, we assume that $x \le y$.

Clearly,

$$\min\{x, y\} = x = \frac{2}{\frac{1}{x} + \frac{1}{x}} \le \frac{2}{\frac{1}{x} + \frac{1}{y}} = H(x, y)\,.$$

Moreover, $\min\{x, y\} = H(x, y)$ if and only if $x = y$. Analogously,

$$\max\{x, y\} = y = \sqrt{\frac{y^2 + y^2}{2}} \ge \sqrt{\frac{x^2 + y^2}{2}} = Q(x, y)$$

and $\max\{x, y\} = Q(x, y)$ if and only if $x = y$.

Here is one relatively straightforward proof of the *AM–GM inequality*

$$G(x, y) = \sqrt{xy} \le \frac{1}{2}(x + y) = A(x, y)\,.$$

Consider a rectangle with side lengths x and y. We then create three additional copies of it and arrange them as shown in Figure 5 so that a square of side $x + y$ is formed. The area of the square is $(x + y)^2$ while the total area of the rectangles is $4xy$.

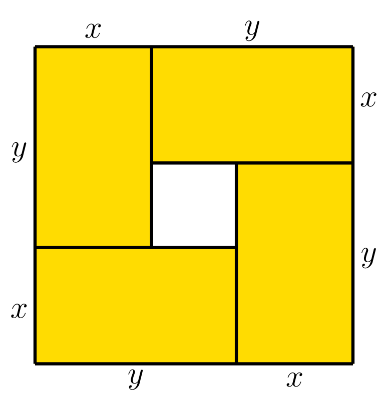

Figure 5: Arrangement of the four rectangles of side length x and y

From Figure 5, it follows that $4xy \le (x + y)^2$ which, by taking square roots of both sides and dividing both sides by 2, implies

$$\sqrt{xy} \le \frac{1}{2}(x + y)\,.$$

The equality $4xy = (x+y)^2$ of the two areas clearly holds if and only if the values of x and y are equal.

Another proof of this AM–GM inequality can be demonstrated using the semicircle k that is known to us. Furthermore, we can derive all inequalities between the four means. It is enough if we combine Figures 1, 2, 3 and 4 into one. This is illustrated in Figure 6.

The length $|PR| = H(x,y)$ is never bigger than the length $|PN| = G(x,y)$, since PN is the hypotenuse of the triangle PNR. This line segment PN is never longer than the radius $|MO| = A(x,y)$ of the semicircle k. Furthermore, $|MO|$ is not bigger than the length $|NO| = Q(x,y)$ since NO is the hypotenuse of the triangle NMO. Therefore, the inequalities $H(x,y) \leq G(x,y) \leq A(x,y) \leq Q(x,y)$ hold.

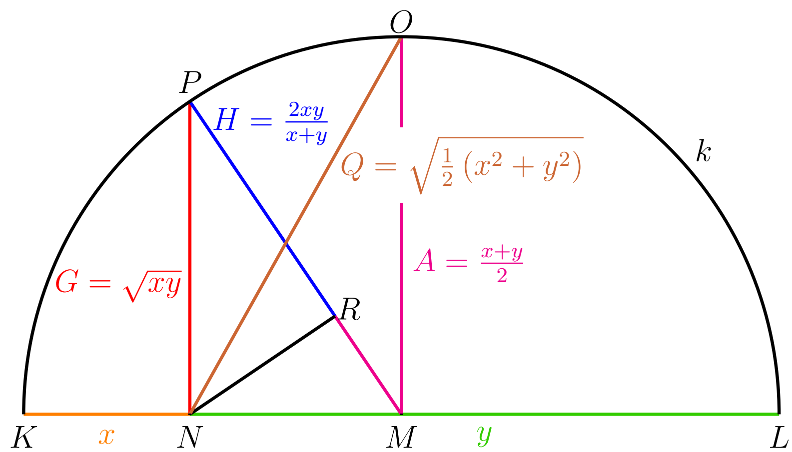

Figure 6: All four means represented in one picture

Figure 7 illustrates Figure 6 for the special case $x = y$. In this case, the points M, N, and R coincide; so do O and P.

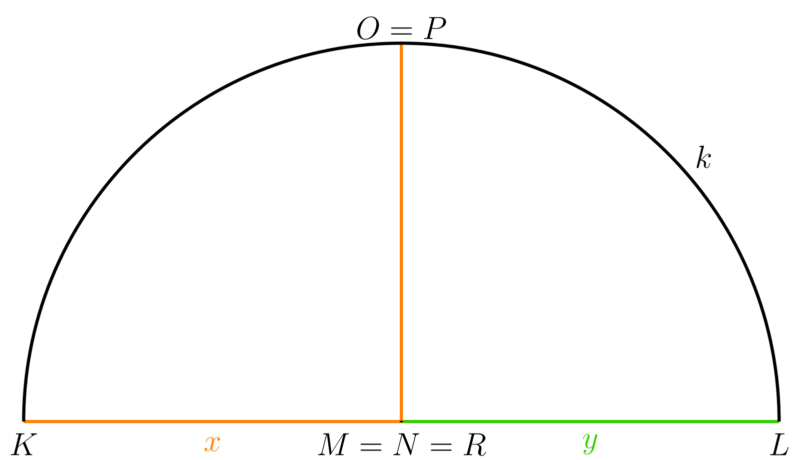

Figure 7: The special case $x = y$

Therefore, we have

$$|RP| = |NP| = |MO| = |NO|$$

which implies that

$$H(x,y) = G(x,y) = A(x,y) = Q(x,y)\,.$$

Equalities clearly occur if and only if $x = y$.

References

[1] D. Djukić, V. Janković, I. Matić, N. Petrović, *The IMO Compendium: A Collection of Problems Suggested for The International Mathematical Olympiads: 1959-2009 Second Edition*, Springer Science & Business Media, 2011.