Parabola Volume 61, Issue 1 (2025)

An introduction to Taylor series and their applications: from proving Euler's formula to computing π

Marvin Hicke[1]

1 Introduction

In a nutshell, a Taylor series decomposes a function $f(x)$ into an infinite series, with each term involving a power of x and a coefficient determined by the function's derivatives at a specific point $x = a$.

Taylor series are widely used in mathematics, physics, engineering and other fields for approximating functions and solving differential equations[2]. This paper presents a concise yet intuitive introduction to Taylor series. We show how the series naturally emerge from applying the Fundamental Theorem of Calculus. To demonstrate their utility, we use Taylor series to develop numerical methods for approximating the values of e and π, prove the famous Euler formula, and explore a generalization where single-valued arguments are replaced by square matrices.

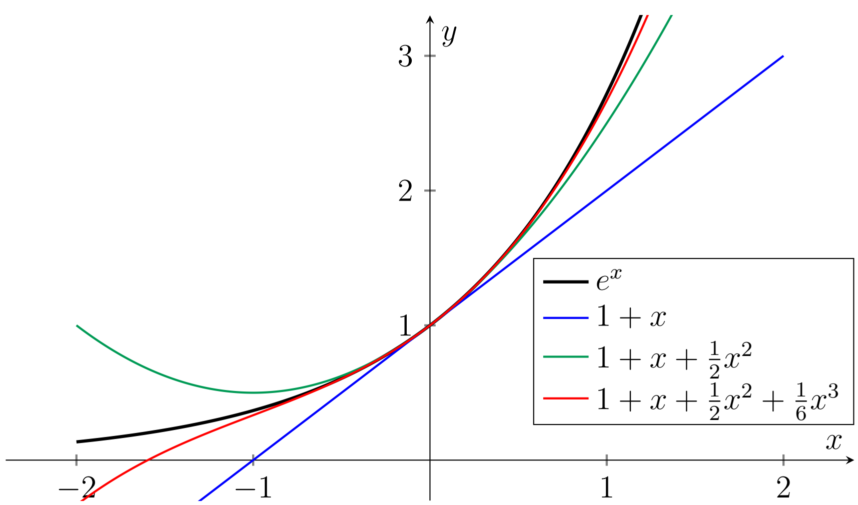

Figure 1: Approximating e^x with polynomials

To motivate the analysis, consider the graph in Figure 1. The exponential function e^x can be approximated with increasing accuracy using linear, quadratic and cubic

[1]Marvin Hicke is a junior at Webb School of Knoxville, Knoxville, TN, USA.

[2]Differential equations are equations where the unknown quantities are functions and their derivatives, e.g., $f(x) = \frac{d}{dx} f(x)$.

polynomials [1]. The linear approximation, constructed as the tangent line to e^x at $x = 0$, matches both the value and slope of e^x at this point.

While the tangent line provides a reasonable approximation for small values of x, the quadratic and cubic polynomials improve accuracy by additionally matching the second and third derivatives at $x = 0$. This illustrates the principle behind Taylor series: systematically adding higher-order terms to the polynomial allows for increasingly accurate approximations. In this sense, Taylor series generalise the tangent line approximation to higher orders.

2 Basic concepts

2.1 Definition of Taylor series

The basic idea behind Taylor series is to approximate functions using polynomials. The expression $T_N(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_N(x-a)^N$ is called the *N-th order Taylor polynomial* of $f(x)$ centred at $x = a$ if the coefficients are chosen such that, at the point $x = a$, the values and the first N derivatives of the function $f(x)$ and the polynomial match.

Generalising this concept to a power series with infinitely many terms leads to the definition of the Taylor series. The Taylor series of a function $f(x)$, which is infinitely differentiable at a point a, is the following power series [1, 2, 3, 4, 5]:

$$\begin{aligned} T(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3 + \cdots . \end{aligned}$$

Here, $n!$ denotes the factorial of n, and $f^{(n)}(a)$ denotes the n-th derivative of f evaluated at $x = a$. The derivative of order zero of f is defined as f itself.

A straightforward differentiation confirms that the derivatives of $T(x)$ and $f(x)$ match at $x = a$; i.e., $T^{(n)}(a) = f^{(n)}(a)$:

$$\begin{aligned} T^{(n)}(x) &= \frac{d^n}{dx^n} T(x) \\ &= \frac{d^n}{dx^n} \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \\ &= \frac{d^n}{dx^n} \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{d^n}{dx^n} \frac{f^{(n)}(a)}{n!}(x-a)^n + \sum_{k=n+1}^{\infty} \frac{d^n}{dx^n} \frac{f^{(k)}(a)}{k!}(x-a)^k \\ &= 0 + \frac{n!}{n!} f^n(a)(x-a)^0 + \sum_{k=n+1}^{\infty} \frac{k!}{k!(k-n)!}(x-a)^{k-n} . \end{aligned}$$

Inserting $x = a$ finally yields

$$T^{(n)}(a) = 0 + \frac{n!}{n!} f^n(a)(a-a)^0 + 0 = f^{(n)}(a)\,.$$

In the following, we focus on the special case where $a = 0$. With that, the series simplifies to

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \cdots\,.$$

Taylor series that are centred at $a = 0$ are also called *Maclaurin series*.

Truncating the series, i.e., including only terms up to order N, yields the N-th order Taylor (Maclaurin) polynomial, which is often called the *N-th order approximation* of $f(x)$:

$$T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} x^n\,.$$

Approximating a function f around a point a by calculating the first few terms of its Taylor series is often called *Taylor expansion*:

$$f(x) = T_N(x) + \mathcal{O}(x^{N+1})\,,$$

where $\mathcal{O}$ is the error in truncating the series; $\mathcal{O}(x^{N+1})$ means that the error is of order $N+1$ or higher in x.

It can be shown that many functions can be approximated with increasing accuracy by Taylor polynomials as higher-order terms are included. However, this approximation is often valid only within a finite interval of convergence, centred at the point a, with R denoting the radius of convergence:

$$\lim_{N\to\infty} T_N(x) = f(x), \quad |x-a| < R\,.$$

Functions satisfying this condition, i.e., functions that can be expanded by a Taylor series, are called analytic functions. Proving that a function f is analytic involves verifying that f is infinitely differentiable at each point over the specified interval, constructing its Taylor series, and demonstrating convergence using techniques like the ratio test.[3]

Remarkably, the property of analyticity allows a function's global behaviour to be fully reconstructed from its local curvature at a single point. However, this fact appears less astounding when one adopts an approach where an analytic function, such as e^x or $\sin(x)$, is defined via its power series representation, and equivalence with geometric, algebraic or other representations is subsequently demonstrated, as is often done in more advanced mathematical treatments (e.g., see [1, 2]).

[3]The ratio test compares terms of the Taylor series to those of a geometric series. If the ratio of consecutive terms is less than 1, then the series converges.

2.2 The error term

Understanding the error term of a Taylor series is crucial, as it provides insight into the rate of convergence of the series. A central question arises: How many terms must be retained in a truncated Taylor series to ensure that the error remains below a desired threshold over a given interval of x-values?

To deepen our understanding of the error term, we explore an alternative and instructive approach to introducing Taylor series, following the ideas in [6, 7, 8]. This method relies on repeated applications of the Fundamental Theorem of Calculus[4]. We start by expressing $f(x)$ as the integral of its derivative over the interval from a to x. This process is then extended to the derivative itself, writing $f'(x)$ as the integral of $f''(t)$, and so forth:

$$f(x) = f(a) + \int_a^x f'(t_1)dt_1 \,.$$

Then f' can now be expressed in terms of f'':

$$f'(t_1) = f'(a) + \int_a^{t_1} f''(t_2)dt_2 \,,$$

yielding

$$\begin{aligned} f(x) &= f(a) + \int_a^x \left(f'(a) + \int_a^{t_1} f''(t_2)dt_2 \right) dt_1 \\ &= f(a) + f'(a)(x-a) + \int_a^x \left(\int_a^{t_1} f''(t_2)dt_2 \right) dt_1 \,, \end{aligned}$$

where we could easily integrate the constant term $f'(a)$ under the integral. We continue by expressing f'' in terms of f''':

$$\begin{aligned} f(x) &= f(a) + f'(a)(x-a) + \int_a^x \int_a^{t_1} \left(f''(a) + \int_a^{t_2} f'''(t_3)dt_3 \right) dt_2\, dt_1 \\ &= f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2} + \int_a^x \int_a^{t_1} \int_a^{t_2} f'''(t_3)\, dt_3\, dt_2\, dt_1 \,. \end{aligned}$$

We can now already recognize that iterative application of the Fundamental Theorem of Calculus to $f(x)$ generates the terms of its Taylor series. Following this pattern up to N-th order in $(x-a)$, we obtain:

$$\begin{aligned} f(x) = f(a) &+ f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2} + \cdots + f^{(N)}(a)\frac{(x-a)^N}{N!} \\ &+ \int_a^x \int_a^{t_1} \int_a^{t_2} \cdots \int_a^{t_N} f^{(N+1)}(t_{N+1})\, dt_{N+1} \cdots dt_3\, dt_2\, dt_1 \,. \end{aligned}$$

[4]The Fundamental Theorem of Calculus states that if $F(x)$ is an antiderivative of a function $f(x)$ on an interval $[a, b]$, then $\int_a^b f(t)\,dt = F(b) - F(a)$. It establishes a connection between differentiation and integration, allowing one to compute definite integrals using antiderivatives.

We can write this as $f(x) = T_N(x) + R_{N+1}(x)$, with R_{N+1} being the error term (also called the *remainder term*). The error term appears to be complicated and not very useful as it has the form of a multiple integral. Yet, through reversing the order of integration, we are able to reduce the multiple integral to a single integral. The key idea here is that it shouldn't matter if we first integrate over t_{N+1} or over t_1 instead, as long as we preserve the geometry of the region of integration, given by

$$a \leq t_{N+1} \leq t_N \leq \cdots \leq t_2 \leq t_1 \leq x \,.$$

That means, when changing the order of integration, we need to be careful in also adjusting the limits of integration accordingly:

$$\begin{aligned} R_{N+1}(x) &= \int_a^x \int_a^{t_1} \int_a^{t_2} \cdots \int_a^{t_N} f^{(N+1)}(t_{N+1})\, dt_{N+1} \cdots dt_3\, dt_2\, dt_1 \\ &= \int_a^x \int_{t_{N+1}}^x \cdots \int_{t_3}^x \int_{t_2}^x f^{(N+1)}(t_{N+1})\, dt_1\, dt_2\, dt_3 \cdots dt_{N+1} \,. \end{aligned}$$

It now becomes clearer why changing the order of integration helps: this is becasue the term $f^{(N+1)}(t_{N+1})$ is a constant with respect to integration over $t_1, t_2, \ldots, t_N$. We thus obtain

$$R_{N+1}(x) = \int_a^x f^{(N+1)}(t_{N+1}) \frac{(x - t_{N+1})^N}{N!} dt_{N+1} \,,$$

and, finally, when dropping the index in the integration variable:

$$R_{N+1}(x) = \frac{1}{N!} \int_a^x f^{(N+1)}(t)(x-t)^N \, dt \,.$$

This result, known as the integral form of the remainder term, naturally arises from recursively expressing $f(x)$ and its derivatives through the Fundamental Theorem of Calculus. Notably, most textbooks derive this term using a different approach, employing a less intuitive method that relies on integration by parts to iteratively transform $\int_a^x f'(t)dt$, which may obscure the connection to the Fundamental Theorem [2].

3 Calculating Maclaurin series of various functions

In this section, we explore the Taylor series of various elementary and composite functions, focusing on the Maclaurin case, where the expansion is centred at $a = 0$.

3.1 Polynomials

We begin with polynomials, which represent a straightforward case in the context of Taylor series. Consider the polynomial $f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$. The Taylor expansion is

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots \,.$$

The derivatives of $f(x)$ evaluated at $x = 0$ are:

$$f(0) = c_0, \quad f'(0) = c_1, \quad f''(0) = 2!c_2, \quad f^{(3)}(0) = 3!c_3, \quad \ldots, \quad f^{(n)}(0) = n!c_n\,.$$

Since the polynomial is finite and the derivatives of $f(x)$ beyond its degree n are zero, this simplifies to

$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n\,.$$

This result confirms that the Maclaurin series (and in fact any Taylor series) for a polynomial is identical to the polynomial itself. Unlike most Taylor series, which are infinite, the series for a polynomial terminates after the n-th term because all higher-order derivatives vanish.

3.2 The trigonometric function $\sin(x)$

To calculate the Maclaurin series of the sine function, we make use of the cyclical feature of repeated differentiation of $\sin(x)$:

$$\begin{aligned}
\sin(0) &= 0\,;\\
\sin'(0) &= \cos(0) = 1\,;\\
\sin''(0) &= -\sin(0) = 0\,;\\
\sin^{(3)}(0) &= -\cos(0) = -1\,;\\
\sin^{(4)}(0) &= \sin(0) = 0\,.
\end{aligned}$$

Plugging in the values of the derivatives of $\sin(x)$ evaluated at $x = 0$ into the Taylor series formula yields

$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots\,.$$

Following this pattern, we see that the signs for odd power terms alternate. On the other hand, even powers of x are skipped because evaluation of $\sin(x)$ at $x = 0$ results in 0. We can express this result using summation notation:

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}\,.$$

To test if and where the series converges, we apply the ratio test: $\left|\frac{a_{n+1}}{a_n}\right| < 1$ for large enough n, where a_n and a_{n+1} are two consecutive terms of the series:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{x^{2(n+1)+1}}{(2(n+1)+1)!}}{\frac{x^{2n+1}}{(2n+1)!}} = \frac{x^2}{(2n+3)(2n+2)}\,.$$

For large n, the term $(2n + 3)(2n + 2)$ grows without bound, ensuring that the ratio approaches 0 for any finite value of x. Thus, the series converges for all x, meaning

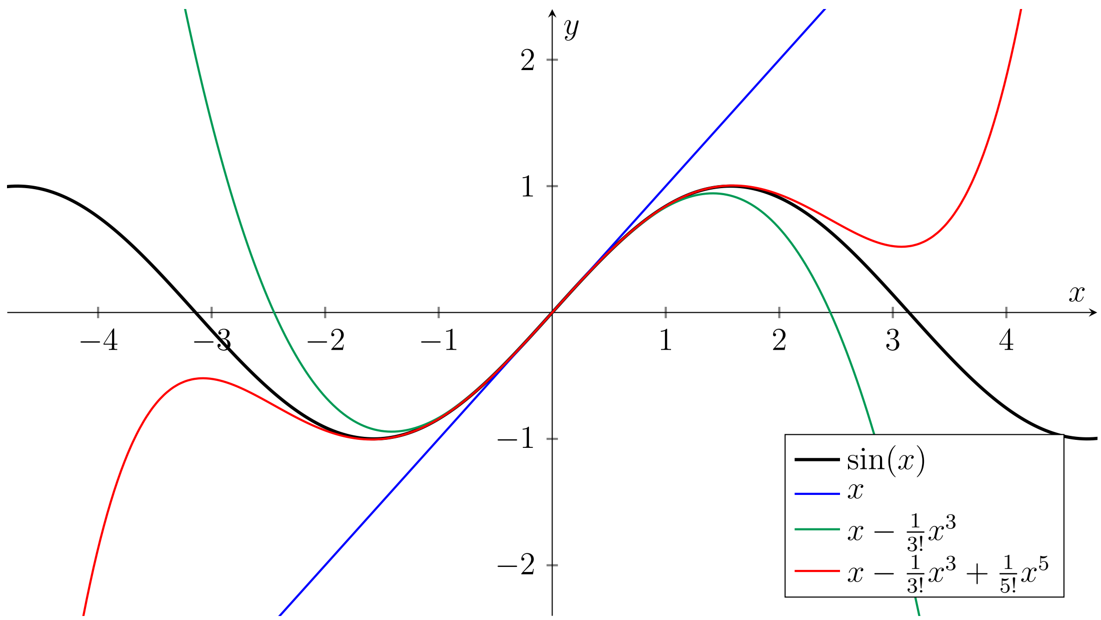

Figure 2: Approximating $\sin(x)$ by Taylor polynomials.

the Maclaurin series for $\sin(x)$ has an infinite radius of convergence. To confirm that the series not just simply converges but actually converges to $\sin(x)$, we examine the remainder term $R_{N+1}(x)$. Using the integral form of the remainder from before, we have

$$R_{N+1}(x) = \frac{1}{N!} \int_0^x f^{(N+1)}(t)(x-t)^N \, dt \,,$$

where $f^{(N+1)}(t)$ is the $(N+1)$-th derivative of $\sin(x)$, which alternates between $\pm\sin(t)$ and $\pm\cos(t)$. Since $|\sin(t)| \leq 1$ and $|\cos(t)| \leq 1$, we can bound $|f^{(N+1)}(t)|$ by 1. Thus,

$$|R_{N+1}(x)| \leq \frac{1}{N!} \int_0^x |x-t|^N \, dt = \frac{|x^{N+1}|}{N!(N+1)} = \frac{|x^{N+1}|}{(N+1)!} \,.$$

For fixed x, this upper bound decreases rapidly as N grows, due to the factorial in the denominator. Hence, $|R_{N+1}(x)| \to 0$ as $N \to \infty$, confirming that the Maclaurin series for $\sin(x)$ not only converges but does so to the original function $\sin(x)$. This result aligns with the well-known fact that $\sin(x)$ is an analytic function defined for all real numbers.

As an example, we determine up to which power of x the Taylor polynomial needs to include terms in order to achieve an error of less than 0.01 for $|x| \leq 5$. Numerically, we find that $\frac{5^{13+1}}{(13+1)!} \approx 0.02$, while $\frac{5^{14+1}}{(14+1)!} \approx 0.007$. This indicates that the required Taylor polynomial is $x - \frac{1}{3!}x^3 + \cdots + \frac{1}{13!}x^{13} - \frac{1}{15!}x^{15}$, as including terms up to x^{15} ensures the error falls below the desired threshold.

3.3 The trigonometric function $\cos(x)$

To calculate the Maclaurin series for the cosine function, we again apply the general formula and insert the values of the derivatives of $\cos(x)$ evaluated at $x = 0$:

$$\begin{aligned}
\cos(0) &= 1, \\
\cos'(0) &= -\sin(0) = 0, \\
\cos''(0) &= -\cos(0) = -1, \\
\cos^{(3)}(0) &= \sin(0) = 0, \\
\cos^{(4)}(0) &= \cos(0) = 1.
\end{aligned}$$

Hence,

$$\begin{aligned}
\cos(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots \\
&= 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots
\end{aligned}$$

Following this pattern, we see that the Taylor series for the cosine function skips the odd powers. Using the summation notation, we obtain

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \, .$$

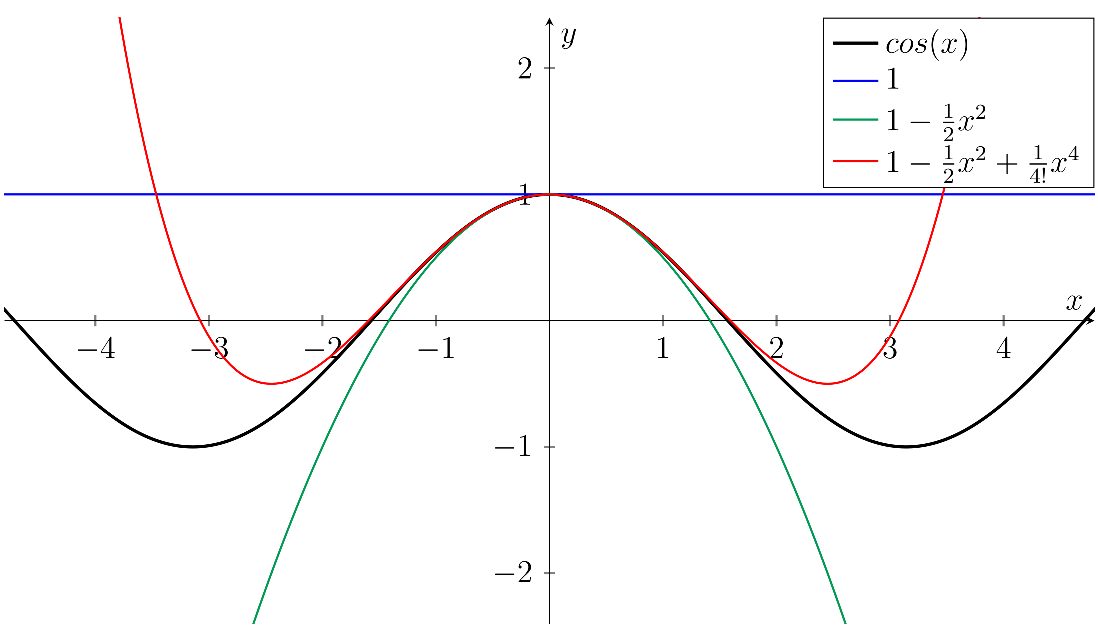

Figure 3: Approximating $\cos(x)$ by Taylor polynomials.

An alternative way to determine the Taylor expansion for $\cos(x)$ is to simply calculate the derivative of the series expansion for $\sin(x)$. This works because $\cos(x)$ is the derivative of $\sin(x)$:

$$\begin{aligned}
\cos(x) &= \frac{d}{dx}\sin(x) \\
&= \frac{d}{dx}\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots\right) \\
&= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots ,
\end{aligned}$$

which is the same result as above.

3.4 The exponential function e^x

To calculate the Taylor series for $f(x) = e^x$, we again need to calculate the derivatives at $x = 0$. Because $\frac{d}{dx}e^x = e^x$, all derivatives evaluated at $x = 0$ are 1. Plugging this into the Maclaurin series formula yields

$$\begin{aligned}
f(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots \\
&= e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots .
\end{aligned}$$

Following this pattern, we obtain

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} .$$

3.5 The generating function of the geometric series $\frac{1}{1-x}$

Let us now examine the geometric series and its relationship to Taylor series. The geometric series is given by:

$$S = 1 + x + x^2 + x^3 + \cdots .$$

To determine the summation formula of this series, we multiply the series by x:

$$Sx = x + x^2 + x^3 + x^4 + \cdots .$$

Subtracting the second equation from the first gives:

$$S - Sx = 1 .$$

Solving for S, we find:

$$S = \frac{1}{1-x} .$$

This result shows that the geometric series converges to the function $\frac{1}{1-x}$ when $|x| < 1$. Thus, the geometric series is the Taylor expansion of $\frac{1}{1-x}$ at $x = 0$:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots .$$

To verify this, we compute the Taylor series of $\frac{1}{1-x}$ directly. The n-th derivative of $\frac{1}{1-x}$ is

$$\frac{d^n}{dx^n}\frac{1}{1-x} = \frac{n!}{(1-x)^{n+1}} .$$

Evaluating this at $x = 0$, we find:

$$f^{(n)}(0) = n! .$$

Substituting into the Taylor series formula:

$$\begin{aligned}\frac{1}{1-x} &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{n!}{n!} x^n \\ &= \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots .\end{aligned}$$

Therefore, we confirm that the Taylor series of $\frac{1}{1-x}$ matches the geometric series, valid for $|x| < 1$.

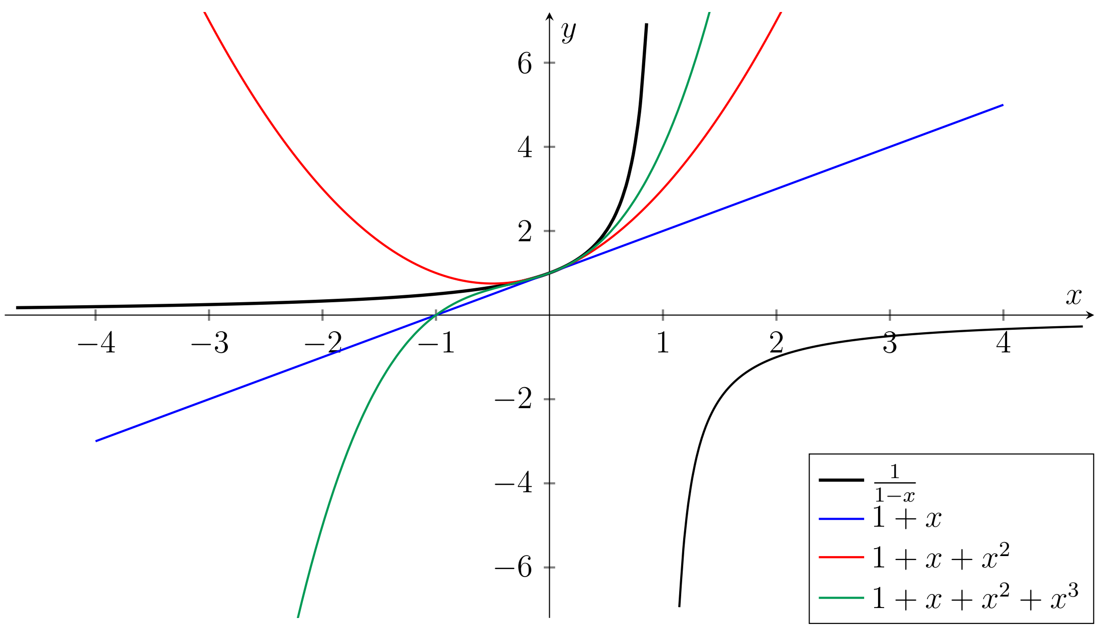

Figure 4: Approximating $\frac{1}{1-x}$ by Taylor polynomials.

3.6 The natural logarithm $\ln(x+1)$

When deriving the Taylor series for logarithmic functions, it is more convenient to focus on $\ln(1+x)$ rather than $\ln(x)$ for two key reasons. First, $\ln(x)$ is not defined at $x=0$, making it unsuitable for centering a Taylor series at this point. Second, $\ln(1+x)$ has a well-behaved derivative, $\frac{1}{1+x}$, which is directly related to the geometric series. This connection simplifies the derivation of its Taylor series.

To derive the Taylor series for $\ln(1+x)$, we start with its derivative:

$$\frac{d}{dx}\ln(1+x) = \frac{1}{1+x}\,.$$

We recall from the geometric series that:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad \text{for } |x|<1\,.$$

Substituting x with $-x$, we obtain:

$$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n, \quad \text{for } |x|<1\,.$$

Using this, we express $\ln(1+x)$ as the integral of its derivative:

$$\ln(1+x) = \int_0^x \frac{1}{1+t}\,dt = \int_0^x \sum_{n=0}^{\infty}(-1)^n t^n\,dt\,.$$

Since the series converges uniformly for $|x|<1$, we can exchange the sum and the integral:

$$\ln(1+x) = \sum_{n=0}^{\infty}(-1)^n \int_0^x t^n\,dt\,.$$

Using the power rule for integration, we compute:

$$\int_0^x t^n\,dt = \frac{x^{n+1}}{n+1}\,.$$

Substituting this result, we find the Taylor series for $\ln(1+x)$:

$$\begin{aligned}\ln(1+x) &= \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1} \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots, \quad \text{for } |x|<1\,.\end{aligned}$$

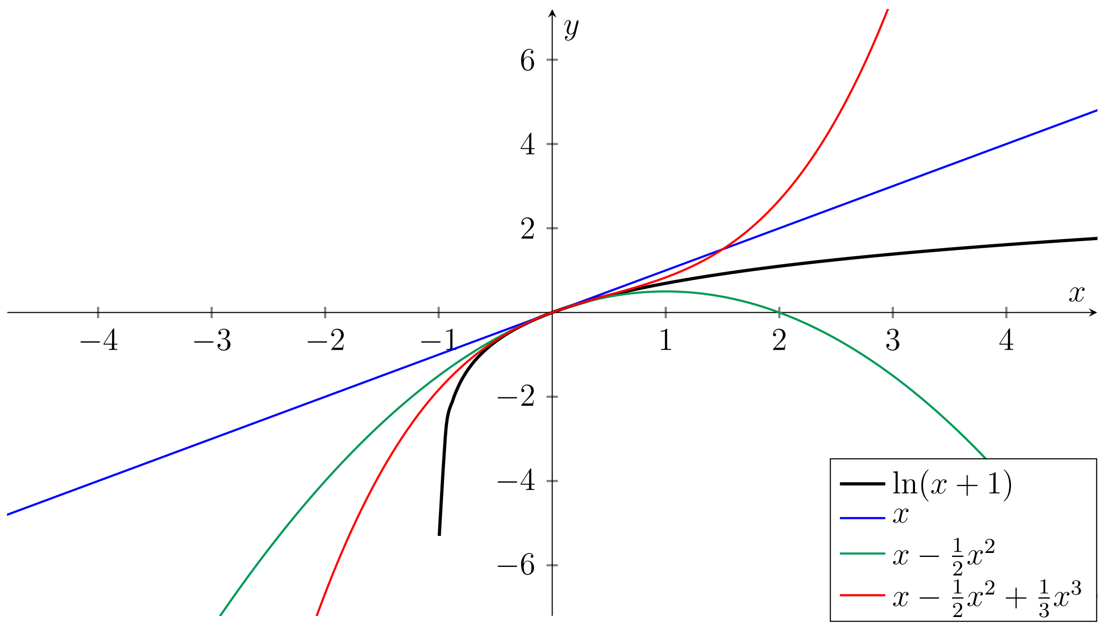

Figure 5: Approximating $\ln(x+1)$ by Taylor polynomials.

3.7 The inverse trigonometric function $\arctan(x)$

To derive the Taylor series for $\arctan(x)$, we start with its derivative:

$$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}\,.$$

Recall from the geometric series that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots\,, \quad \text{for } |x| < 1\,.$$

Substituting x^2 for x, we write

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots\,, \quad \text{for } |x| < 1\,.$$

Now, integrating term by term, we obtain the Taylor series for $\arctan(x)$:

$$\begin{aligned}\arctan(x) &= \int \frac{1}{1+x^2}\,dx = \int \left(1 - x^2 + x^4 - x^6 + \cdots\right) dx \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\,, \quad \text{for } |x| < 1\,.\end{aligned}$$

In summation notation, the Taylor series for $\arctan(x)$ is

$$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \quad \text{for } |x| < 1\,.$$

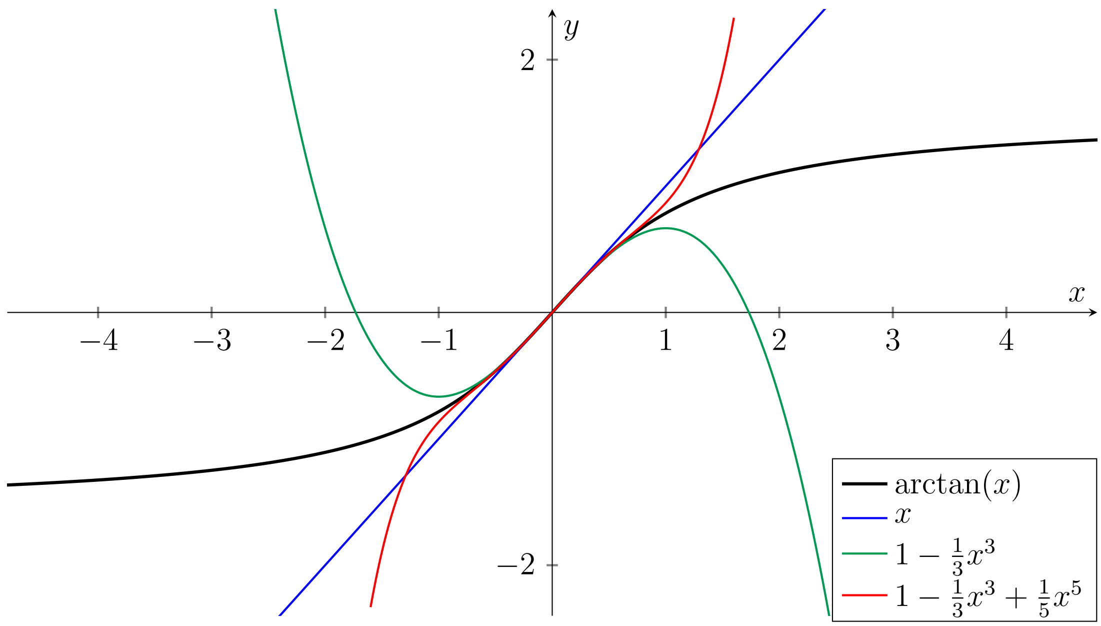

Figure 6: Approximating $\arctan(x)$ by Taylor polynomials.

3.8 Composed functions and products of functions

Now that we have established Taylor series expansions for elementary functions, we turn to more complex cases, such as composed functions and products of functions.

Composed functions
For a function of the form $f(g(x))$, we first expand $f(u)$ as a Taylor series around $u = 0$:

$$f(u) = f(0) + f'(0)u + \frac{f''(0)}{2}u^2 + \frac{f'''(0)}{6}u^3 + \cdots .$$

Next, we substitute $u = g(x)$ into this series. If $g(x)$ itself is not a simple polynomial, we may also need to expand $g(x)$ as a Taylor series around $x = 0$. After substitution, terms involving powers of x from $g(x)$ are combined with the coefficients from $f(u)$ to collect terms of the same order in x. Simplifying these terms yields the Taylor series for $f(g(x))$.

Example 1: Expansion of $\sin(x^2)$. Consider the composed function $\sin(x^2)$. Substituting x^2 into the Taylor series for $\sin(x)$, we have:

$$\sin(x^2) = x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!} + \cdots .$$

Simplifying each term gives

$$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots .$$

Example 2: Expansion of $\cos(\sin(x))$. Next, consider $\cos(\sin(x))$. Substituting the Taylor series for $\sin(x)$ into the expansion for $\cos(x)$, we write:

$$\cos(\sin(x)) = 1 - \frac{1}{2}(\sin(x))^2 + \frac{1}{4!}(\sin(x))^4 - \cdots .$$

Using the Taylor series for $\sin(x)$, we get

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots ,$$

so, by substituting,

$$\cos(\sin(x)) = 1 - \frac{1}{2}(x - \frac{x^3}{3!} + \cdots)^2 + \frac{1}{4!}(x - \frac{x^3}{3!} + \cdots)^4 - \cdots .$$

Expanding the squares and higher powers, and combining like terms, we find that

$$\cos(\sin(x)) = 1 - \frac{x^2}{2} + \frac{5x^4}{24} - \cdots .$$

Products of functions

Now we consider the Taylor series for the product of two functions, $f(x) \cdot g(x)$. To compute the product, we multiply the Taylor expansions of $f(x)$ and $g(x)$, combining terms of the same power of x.

Example 3: Expansion of $\sin(x)e^x$. As an example, we compute the Taylor expansion of $\sin(x)e^x$ up to the fourth power in x. The product of the expansions

$$\sin(x) = x - \frac{x^3}{3!} + \cdots , \quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots ,$$

is

$$\sin(x)e^x = \left(x - \frac{x^3}{3!} + \cdots\right)\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) .$$

Combining terms of the same power of x, we obtain

$$\sin(x)e^x = x + x^2 + \frac{x^3}{3} + \mathcal{O}(x^5) .$$

Interestingly, the fourth-order term cancels out in this case.

4 Determining the numerical values of e and π

4.1 Calculating the value of e

The mathematical constant e is one of the most important numbers in mathematics. Its value can be determined numerically using the Taylor series expansion of the exponential function, e^x. Specifically, when $x = 1$, the Taylor series for e^1 converges to e:

$$e = e^1 = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots .$$

The series provides an efficient numerical formula for calculating e. Each additional term significantly improves the accuracy of the approximation. To illustrate the convergence, Table 1 shows the approximate value of e obtained by truncating the series at the N-th term, along with the corresponding error.

N	Approximate value e_N	Error $\lvert e - e_N \rvert$	Percent error
0	1.0000	1.7183	63.21%
1	2.0000	0.7183	26.42%
2	2.5000	0.2183	8.03%
3	2.6667	0.0516	1.90%
4	2.7083	0.0099	0.36%
5	2.7167	0.0016	0.06%
6	2.7181	0.0002	0.01%
$\vdots$	$\vdots$	$\vdots$	$\vdots$
∞	2.7183	0	0

Table 1: Approximation of e using the Taylor series of e^1.

4.2 Calculating the value of $\frac{\pi}{4}$

Similarly to e, the constant π is one of the most important numbers in mathematics. Its precise value has fascinated mathematicians for centuries, and numerous methods have been developed to calculate it. Here, we use the Taylor expansion of the inverse tangent function, $\arctan(x)$, to approximate π, using the fact that $\tan\left(\frac{\pi}{4}\right) = 1$.

The Leibnitz Formula

As we derived earlier, the Taylor series for $\arctan(x)$ is

$$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \quad \text{for } |x| \leq 1\,.$$

Setting $x = 1$, this series converges to $\frac{\pi}{4}$:

$$\frac{\pi}{4} = \arctan(1) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}\,.$$

This alternating series, known as the Leibniz Formula, provides a simple method to approximate π:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots\,.$$

Although the series converges, its rate of convergence is relatively slow because the terms decrease inversely with the odd integers. Table 2 illustrates the approximate values of π obtained by truncating the series at the N-th term and multiplying by 4.

N	Approximate value π_N	Error $\lvert\pi - \pi_N\rvert$	Percent error
0	4.0000	0.8584	27.32%
1	2.6667	0.4749	15.12%
2	3.4667	0.3251	10.35%
3	2.8952	0.2464	7.84%
4	3.3397	0.1981	6.31%
5	2.9760	0.1656	5.27%
6	3.2837	0.1421	4.52%
$\vdots$	$\vdots$	$\vdots$	$\vdots$
100	3.1515	0.0100	0.32%
$\vdots$	$\vdots$	$\vdots$	$\vdots$
∞	3.1416	0	0

Table 2: Approximation of π using the Taylor series for $\arctan(1)$.

As shown in the table, the Leibniz formula converges to π, but the convergence is significantly slower compared to the exponential series for e. Although it serves as a mathematically simple and intriguing method, alternative series or algorithms are often used for faster convergence when high precision is required.

Machin-like formulas

Interestingly, there is a trick to achieve much faster convergence while still building on the fact that $\arctan(1) = \frac{\pi}{4}$. This is done by expressing $\arctan(1)$ as a sum of $\arctan(a)$ and $\arctan(b)$, with a and b both less than 1, and then expanding these two expressions separately. Such decompositions are called *Machin-like* formulas [9, 10]. The respective Taylor series for $\arctan(a)$ and $\arctan(b)$ converge much faster because their terms decrease exponentially with increasing order N of the Taylor expansion, whereas in the Leibniz formula ($x = 1$), the terms decrease only as reciprocals of odd integers.

To find suitable numbers a and b, we start with the following transformation of the tangent of a sum:

$$\begin{aligned}\tan(\alpha+\beta) &= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} \\ &= \frac{\cos(\alpha)\sin(\beta)+\sin(\alpha)\cos(\beta)}{\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)} = \frac{\frac{\sin(\alpha)}{\cos(\alpha)}+\frac{\sin(\beta)}{\cos(\beta)}}{1-\frac{\sin(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}} = \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}\,.\end{aligned}$$

Substituting $\tan(\alpha)$ with a and $\tan(\beta)$ with b yields

$$\tan(\alpha+\beta) = \frac{a+b}{1-ab}$$

so $\alpha + \beta = \arctan\left(\frac{a+b}{1-ab}\right)$; therefore,

$$\arctan(a) + \arctan(b) = \arctan\left(\frac{a+b}{1-ab}\right)\,.$$

Since we want $\arctan(a) + \arctan(b) = \arctan(1) = \frac{\pi}{4}$, we require $\frac{a+b}{1-ab} = 1$. This formula helps us to find suitable values for a and b. Trying out $a = \frac{1}{2}$ and solving for b,

$$\frac{\frac{1}{2} + b}{1 - \frac{1}{2}b} = 1\,,$$

yields $b = \frac{1}{3}$, and thus

$$\frac{\pi}{4} = \arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right)\,.$$

This result is convenient because both $a = \frac{1}{2}$ and $b = \frac{1}{3}$ are simple fractions, making numerical computation very efficient. The error in the numerical approximation of $\frac{\pi}{4}$ is dominated by the larger argument, here $x = \frac{1}{2}$. For example, if $N = 3$, then the error can be approximated as

$$\left|\frac{\pi}{4} - T_{N=3}\right| \approx \frac{1}{5}\left(\frac{1}{2}\right)^5\,.$$

To achieve smaller errors, smaller values of a and b are required. A famous example is Machin's original formula from 1706:

$$\frac{\pi}{4} = 4\arctan\left(\frac{1}{5}\right) + \arctan\left(\frac{1}{239}\right)\,.$$

Here, the arguments are much smaller, leading to significantly faster convergence. To prove that this formula is correct, we pairwise apply the addition theorem repeatedly showing that the five $\arctan$ terms add up to $\arctan(1)$.

In recent years, even more efficient Machin-like formulas have been discovered. For example, Takano (1982) [9, 10]:

$$\frac{\pi}{4} = 12\arctan\left(\frac{1}{49}\right) + 32\arctan\left(\frac{1}{57}\right) - 5\arctan\left(\frac{1}{239}\right) + 12\arctan\left(\frac{1}{110443}\right)\,.$$

Figure 7 compares the errors of different formulas as a function of the number of terms included in their Taylor expansions. The results demonstrate the dramatic improvement in convergence rates that Machin-like formulas can achieve, making them well suited for high-precision computations of π.

5 Proof of the Euler formula

We will now use the previously derived Taylor series for the exponential, sine and cosine functions to uncover a remarkable relationship between these functions. Although we have not yet introduced Taylor series for complex numbers, we will proceed experimentally by substituting x with ix in the Taylor series for e^x, where i is the imaginary unit [1, 2, 11].

The imaginary unit i is defined by the following property:

$$i^2 = -1.$$

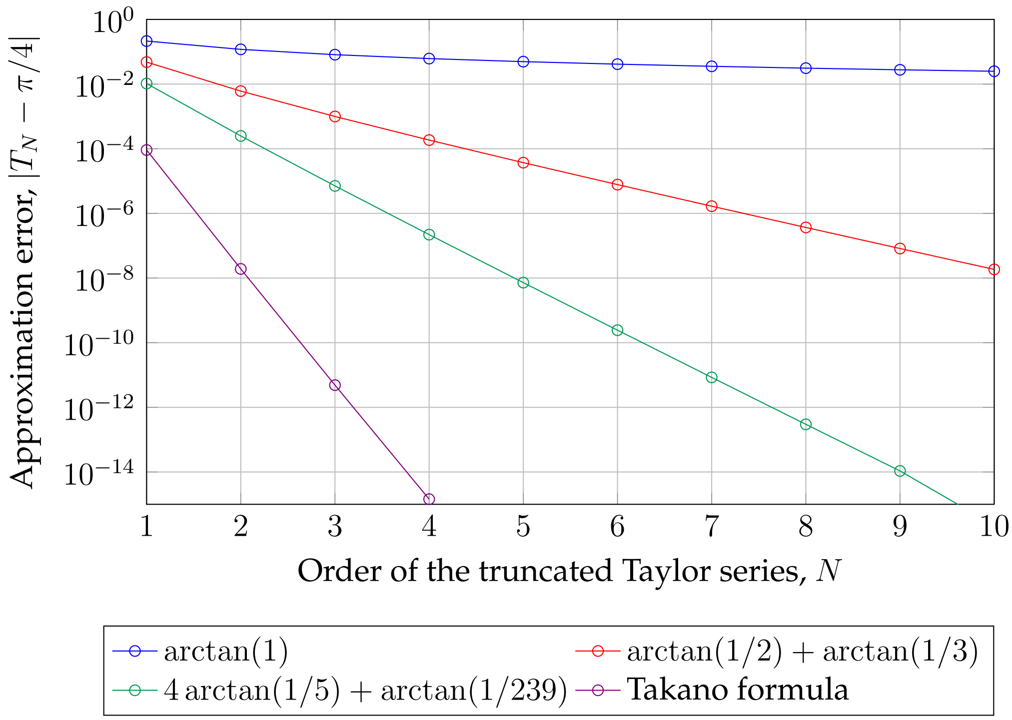

Figure 7: Comparing the error $|T_N - \pi/4|$ of different approximations formulas

Using this, we see that powers of i cycle with a period of four:

$$i^1 = i, \quad i^2 = -1, \quad i^3 = -i, \quad i^4 = 1, \quad i^5 = i, \quad \ldots .$$

Substituting x with ix in the Taylor series for e^x, we obtain:

$$\begin{aligned} e^{ix} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \cdots \\ &= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \cdots . \end{aligned}$$

We group the terms with and without i to separate the real and imaginary parts:

$$e^{ix} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) .$$

The real part matches the Taylor series for $\cos(x)$ and the imaginary part for $\sin(x)$. Thus, we obtain the famous Euler's Formula:

$$e^{ix} = \cos(x) + i\sin(x) .$$

The formula shows that multiplying a complex number by e^{ix} corresponds to a rotation by angle x in the complex plane.

A particularly beautiful result arises when substituting $x = \pi$ into Euler's Formula:

$$e^{i\pi} + 1 = 0\,.$$

This equation, known as Euler's Identity, is celebrated as one of the most elegant equations in mathematics, uniting the constants e, π, 1, 0, and i in a single, simple formula.

6 Generalization to matrices

In this section, we explore a generalization of the Taylor series by replacing the scalar variable x with a square matrix A [11]. This approach allows us to define functions of matrices via Taylor series. This is analogous to the substitution of x with ix in the Euler formula.

For a square matrix A, the matrix function $f(A)$ is defined as:

$$f(A) = f(0)I + f'(0)A + \frac{f''(0)}{2}A^2 + \frac{f'''(0)}{3!}A^3 + \cdots\,,$$

where I is the identity matrix. For a 2×2 matrix, I is

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\,.$$

As an example, we examine the Taylor expansion of the so-called matrix exponential e^A [11, 12]. Substituting x with a square matrix A, the Taylor series for e^x becomes

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots\,.$$

Let A now be the following skew-symmetric matrix:

$$A = \begin{pmatrix} 0 & -\alpha \\ \alpha & 0 \end{pmatrix}\,.$$

Let $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, so that $A = \alpha B$. By substituting into the series, we get

$$e^{\alpha B} = I + \alpha B + \frac{1}{2!}(\alpha B)^2 + \frac{1}{3!}(\alpha B)^3 + \cdots\,.$$

Using the property that $B^2 = -I$, $B^3 = -B$, and $B^4 = I$, the powers of B alternate cyclically:

$$e^{\alpha B} = I + \alpha B - \frac{\alpha^2}{2!}I - \frac{\alpha^3}{3!}B + \frac{\alpha^4}{4!}I + \cdots\,.$$

Grouping terms involving I and B, we have:

$$e^{\alpha B} = I\left(1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \cdots\right) + B\left(\alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} - \cdots\right)\,.$$

We recognize that the terms multiplying I match the Taylor series for $\cos(\alpha)$ and the terms multiplying B match the Taylor series for $\sin(\alpha)$. We thus obtain:

$$e^{\alpha B} = \cos(\alpha)I + \sin(\alpha)B\,.$$

Writing this expression in matrix form, we get

$$e^{\alpha B} = \cos(\alpha)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \sin(\alpha)\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\,.$$

Simplifying this expression gives us the following identity:

$$e^{\alpha B} = \begin{pmatrix}\cos(\alpha) & -\sin(\alpha)\\ \sin(\alpha) & \cos(\alpha)\end{pmatrix}\,.$$

The resulting matrix is the well-known rotation matrix. This result demonstrates that $e^{\alpha B}$, for this particular B, corresponds to a counter-clockwise rotation by angle α in the plane.

This result closely parallels the derivation of Euler's Formula, where we substituted x with ix in e^x. In both cases, the exponential function is extended to new domains – in Euler's Formula, t o complex numbers, and here, to square matrices. The core link between i and B is that each of them "squares to -1". Symbolically, $i^2 = -1$ and $B^2 = -I$. This property indicates that multiplying by i or acting by B implements a 90° rotation in their respective spaces: i in the complex plane and B in $\mathbb{R}^2$. Exponentiating these "rotation generators" yields a continuous family of rotations, e^{ix} or $e^{\alpha B}$, that sweep out the unit circle in the complex plane or the 2D real plane. This illustrates the unifying role of the exponential function in connecting algebraic and geometric structures across different mathematical domains [11].

References

[1] T.M. Apostol, *Calculus, Volume 1*, Second Edition, Wiley, 1991.

[2] M. Spivak, *Calculus*, Fourth Edition, Publish or Perish, 2008.

[3] D. Patrick, *Calculus*, AoPS Incorporated, 2020.

[4] Wikipedia, Taylor Series, https://en.wikipedia.org/wiki/Taylor_series, last accessed on 2025-04-13.

[5] MathWorld, Taylor Series, https://mathworld.wolfram.com/TaylorSeries.html, last accessed on 2025-04-13.

[6] Brilliant.org, Taylor's Theorem (with Lagrange Remainder), https://brilliant.org/wiki/taylors-theorem-with-lagrange-remainder/, last accessed on 2025-04-13.

[7] D. Kountourogiannis, A derivation of Taylor's Formula with integral remainder, *Mathematics Magazine* **76**(3) (2003), 217–219.

[8] O. Sisha, On Taylor's Theorem, *Journal of Research of the National Bureau of Standards* **72B**(1) (1968), 5–6.

[9] Wikipedia, Machin-like formula, https://en.wikipedia.org/wiki/Machin-like_formula, last accessed on 2025-04-13.

[10] MathWorld, Machin-Like Formulas, https://mathworld.wolfram.com/Machin-LikeFormulas.html, last accessed on 2025-04-13.

[11] G. Strang, *Linear Algebra and Its Applications*, Fourth Edition, Cengage Learning, 2006.

[12] Wikipedia, Rotation Matrix, https://en.wikipedia.org/wiki/Rotation_matrix, last accessed on 2025-04-13.