GEOMETRY USING PAINTING

A talk given to high school Level I students on May 31, 1973 by Dr R. Street of Macquarie University.

General mathematical principle: To investigate a complicated situation apply simplifying transformations, investigate the simplified situations, and try to transfer this information back to the original situation.

Consequently mathematicians study the transformations themselves and the properties which they preserve and reflect.

I want to centre attention on plane geometry. The above technique is already familiar here. To prove a theorem about an ellipse in general we can apply a rigid motion (= isometry) to "bring the ellipse to the origin."

An *isometry* is a transformation of the plane which:

(a) takes collinear points to collinear points;
(b) takes parallel lines to parallel lines;
(c) takes perpendicular lines to perpendicular lines;
(d) preserves distances.

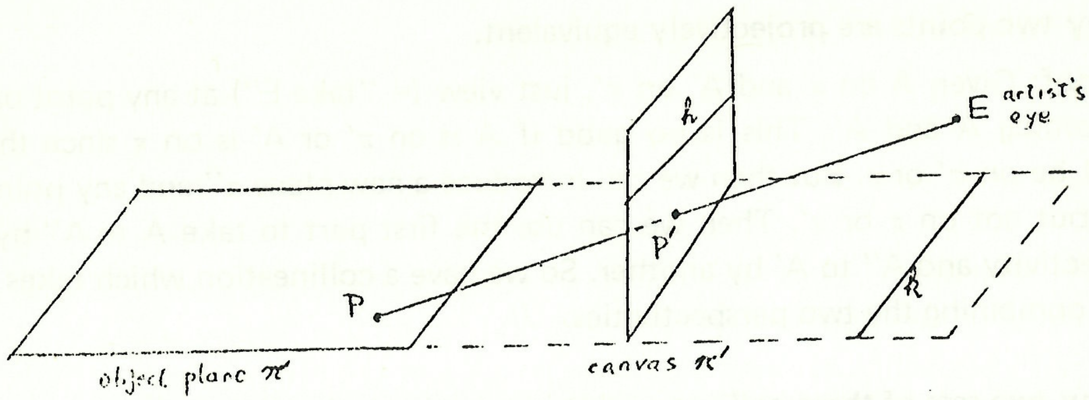
Actually (c) is a consequence of (d) because of Pythagoras. Any isometry is a composite of translations, rotations and reflections. Two figures are said to be *congruent* when there is an isometry which takes one into the other. Area is a property left unchanged by isometries.

A *similarity transformation* is a transformation of the plane which satisfies (a), (b), (c) above. Two figures are called *similar* when there is a similarity transformation which takes one into the other. Angles are left unchanged by these transformations. The ratio of the area of a triangle to the square of the length of a given side is left invariant also.

Isometries and similarity transformations belong to the geometry of Euclid. I wish to discuss transformations which belong to the geometry of artists (Leonardo da Vinci, Albrecht Durer) and mathematicians (Desargues) of the 15th century, called *projective* (or *descriptive) geometry.*

We shall only consider the accurate painting of objects in a plane on a plane canvas.

Each point P of the object plane π gives rise to a point $P' = PE \cap \pi'$ on the canvas π', as long as P is not on the line k of intersection of π and the plane through E parallel to π'. The line k is of no bother to the artist who is not interested in objects behind the canvas, but for our mathematical theory we will consider not



only finite parts of a plane, but rather the whole plane extended infinitely. The similarly defined line on the canvas *is* of interest to the artist. Let h denote the line of intersection of π' and the plane through E parallel to π. This is the line that the artist sees as the "horizon". Parallel lines l, m in π have image lines l',m' in π' which *meet* on h, so that points on h may be thought of as coming from *gradients* of lines in π. This is the well-known phenomenon that railway tracks appear to meet at the horizon.

So we have a transformation from π leaving out k to π' leaving out h which:

(i) takes collinear points to collinear points;

(ii) takes parallel lines to lines meeting on h;

(iii) takes lines meeting on k to parallel lines.

This transformation is called a *perspectivity.* In this definition we include also the case where E is at infinity (i.e. the artist is on another planet to a good approximation). Note that E cannot be on π or π'. A transformation obtained after repeated "painting" (after repeated perspectivities) is called a *collineation.* It turns out that collineations are characterized by properties (i), (ii), (iii).

Algebra fiends might set up coordinate systems in π and π' and show that collineations are determined by a 3x3 non-singular matrix $A = (a_{ij})$ via the rule: P(x,y) goes to P'(x',y') where

$$x' = \frac{a_{11}x + a_{12}y + a_{13}}{a_{31}x + a_{32}y + a_{33}} \qquad y' = \frac{a_{21}x + a_{22}y + a_{23}}{a_{31}x + a_{32}y + a_{33}}$$

The line k is given by the equation $a_{31}x + a_{32}y + a_{33} = 0$; its geometric property is reflected by the algebraic fact that dividing by 0 is naughty. To find the line h the equations must be solved for x and y in terms of x' and y'.

Two figures are called *projectively equivalent* when there is a collineation which takes one into the other. Collineations include the similarity transformations.

I. Any two points are projectively equivalent.

Proof: Given A on π and A′ on π', just view (= "take E") at any point on the line joining A and A′. This is no good if A is on π' or A′ is on π since then E would be on π' or π. But then we can introduce a new plane π'' and any point A″ on it but not on π or π'. Then we can use the first part to take A to A″ by one perspectivity and A″ to A′ by another. So we have a collineation which takes A to A′ on combining the two perspectivities. //

II. Any two sets of three collinear points are projectively equivalent.

Proof: Suppose A,B,C lie on a line l in the plane π and A′,B′,C′ on line l' in π'. Let π_1 be the plane containing A,B,C′; it also contains C. By I, there is a collineation which takes C on π to C′ on π_1. Suppose that A,B on π go to A_1 B_1 on π_1 under this collineation. Then the lines A_1B_1 and A′B′ meet at C′, so the points A_1,B_1,A′,B′ are all in the same plane. So take E to be the point of intersection (may be at infinity) of the lines A_1A′ and B_1B′. Viewing from E takes A_1 to A′, B_1 to B′, C′ to C′. Combining with the first collineation we get a collineation which takes A,B,C to A′,B′,C′ respectively. //

Special degenerate cases can be handled as in I by introducing an extra plane and a perspectivity to put the points in a more general position. For example:

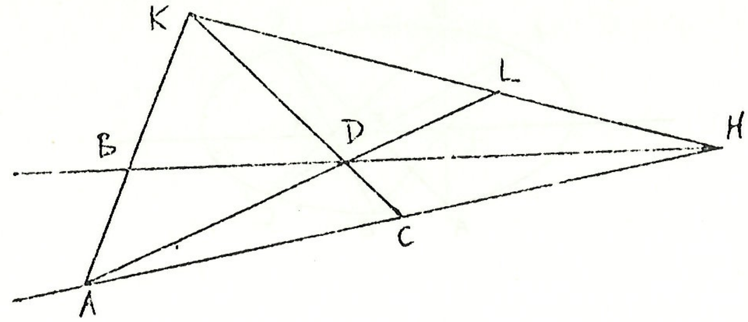
III. In II, one or both of the sets of three collinear points can be replaced by a set of three gradients.

By this I mean that, given three numbers m_1,m_2,m_3 which are to represent gradients of families of lines in a plane π, and three points A_1,A_2,A_3 on a line in a plane π', then there exists a collineation with the line $A_1A_2A_3$ as horizon such that lines in π with gradient m_i correspond to lines in π' through the point A_i (i = 1,2,3,).

Proof: In the plane π choose lines l_1,l_2,l_3 with gradients m_1,m_2,m_3. Apply any perspectivity from π to a plane π_1 with horizon h. The images of l_1,l_2,l_3 meet h at points B_1,B_2,B_3 say. Use II to obtain a collineation from π_1 to π' which takes B_1,B_2,B_3 to A_1,A_2,A_3 respectively. Combining the perspectivity from π to π_1 and the collineation from π_1 to π' we have a collineation from π to π' as required. //

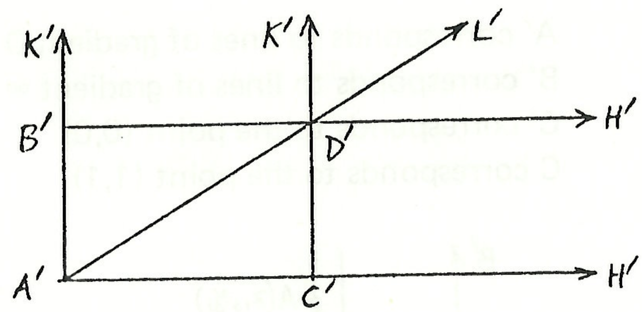
IV. Any two quadrilaterals are projectively equivalent.

Proof: Consider a quadrilateral ABCD.



Let $K = AB \cap CD$, $H = AC \cap BD$, $L = AD \cap HK$.
Consider the collineation with KH as horizon and such that K,L,H correspond to gradients ∞,1,0 (using III).

Image of the quadrilateral



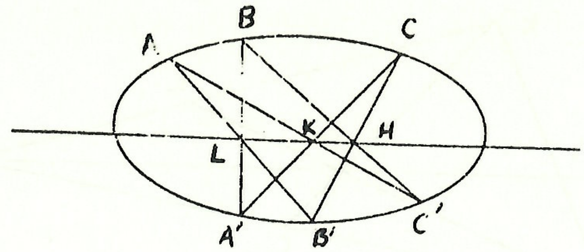
So the quadrilateral becomes a square (the only quadrilaterals with sides gradient $0,\infty$ and diagonal gradient 1). We can transform this square to the unit square with centre the origin by a similarity transformation. So any quadrilateral is projectively equivalent to the unit square centre the origin. //

V. The image of a conic under a collineation is a conic. Any two conics are projectively equivalent.

Proof: A painting of a hyperbola is an ellipse which meets the horizon in two points. A painting of a parabola is an ellipse which meets the horizon in one point. So we need only worry about ellipses. But by viewing an ellipse from the appropriate position we can see a circle. Any circle is similar to the unit circle centre the origin. //

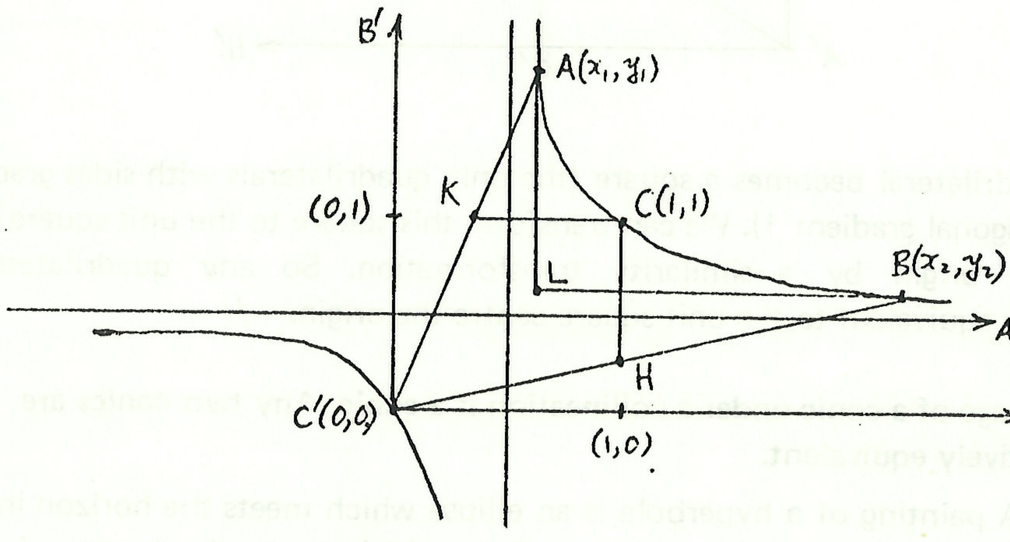
These considerations can be used to give a simple proof of:

Pascal's Mystic Hexagram Theorem: Six points A,B′, C, A′, B, C′ lie on a conic if and only if the three points $L = AB' \cap A'B$, $H = BC' \cap B'C$, $K = CA' \cap C'A$ are collinear.



Proof: Collineations take conics into conics and collinear points to collinear points, so the hypothesis and conclusion of the theorem are not affected by a collineation. Choose a collineation which takes the line A′B′ as horizon in such a way that

A′ corresponds to lines of gradient 0
B′ corresponds to lines of gradient ∞
C′ corresponds to the point (0,0)
C corresponds to the point (1,1)



So without loss of generality we can suppose A′ is at infinity on the x-axis, B′ is at infinity on the y-axis, C′ is (0,0), A is (x_1, y_1), B is (x_2, y_2), C is (1,1). Conics have the form

$$ax^2 + by^2 + c + 2fy + 2gx + 2hxy = 0.$$

For one asymptote to be parallel to the x-axis, we require b = 0, for the other parallel to the y-axis, we require a = 0. For (0,0) to be on it, c = 0. So conic reduces to

$$\frac{f}{x} + \frac{g}{y} + h = 0.$$

In order that this should not degenerate into a pair of lines, $f \neq 0$. So we can divide by f.

$$\frac{1}{x} + \frac{g'}{y} + k = 0.$$

For (1,1) to be on this, $g' = -1-k$.

$$\frac{1}{x} - \frac{(1+k)}{y} + k = 0.$$

We prefer to write this as

$$k = \frac{x-y}{x(y-1)}.$$

$A(x_1,y_1)$, $B(x_2,y_2)$ are on this conic when

$$\frac{x_1-y_1}{x_1(y_1-1)} = \frac{x_2-y_2}{x_2(y_2-1)} = k.$$

So the condition that A,B′,C,A′,B,C′ should be on the one conic is:

$$\frac{x_1-y_1}{x_1(y_1-1)} = \frac{x_2-y_2}{x_2(y_2-1)}$$

H,K,L are the points $(1,y_2/x_2)$, $(x_1/y_1,1)$, (x_1,y_2). These lie on the same line precisely when

$$\frac{(y_2/x_2)-1}{1-(x_1/y_1)} = \frac{y_2-1}{x_1-(x_1/y_1)};$$

that is, $\dfrac{(y_2-x_2)}{x_2(y_1-x_1)} = \dfrac{(y_2-1)}{x_1(y_1-1)}$ as above. //

Pascal's theorem can be used to find new points on a conic when five points are known:

Given A,B,C,D,E, we let $K = AB \cap DE$. Take any H on the line CD. Put $L = HK \cap CB$. Then $F = AH \cap EL$ is on the conic. Indeed, as H runs along the line CD, F runs around the conic.

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