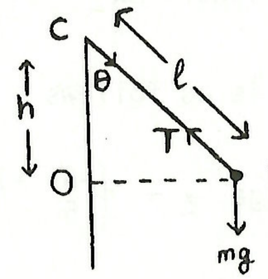
H.S.C. CORNER BY TREVOR

THE CONICAL PENDULUM

In 1981 and 1982 there were some 4 unit questions on the conical pendulum. A particle P of mass m, attached by a light rod to a pivot point C, moves with constant speed in a horizontal circle whose centre O is distant h metres below C, as in the figure.



As the particle moves, the rod PC describes a cone of semi-vertical angle θ, so that $h = \ell \cos\theta$, where ℓ is the length of the rod. The radius of the circle is $OP = h\tan\theta$. Let the angular velocity be ω.

There are two forces acting on P. These are gravity mg, acting vertically downwards, and the tension T in the rod acting along the direction of the rod. We note that Newton's Law $\underline{F} = m\underline{a}$ is a vector equation, and we take horizontal and vertical components of this vector equation. Since there is no vertical acceleration, the vertical forces must sum to zero, to give

$$T\cos\theta = mg. \qquad (1)$$

[Note that the rod makes an angle θ with the vertical, so that the vertical component of the tension in the rod is $T\cos\theta$ upwards].

The horizontal acceleration is $r\omega^2$, in the direction PO, where r is the radius of the circle, i.e., $r = h\tan\theta$. Thus the horizontal component of the tension must provide this horizontal acceleration, and, therefore, according to Newton's Law

$$T\cos(\pi/2 - \theta) = mh\,\omega^2\tan\theta.$$

Noting that $\cos(\pi/2 - \theta) = \sin\theta$, we find that

$$T\cos\theta = mh\omega^2. \qquad (2)$$

Combining (1) and (2) yields the relation

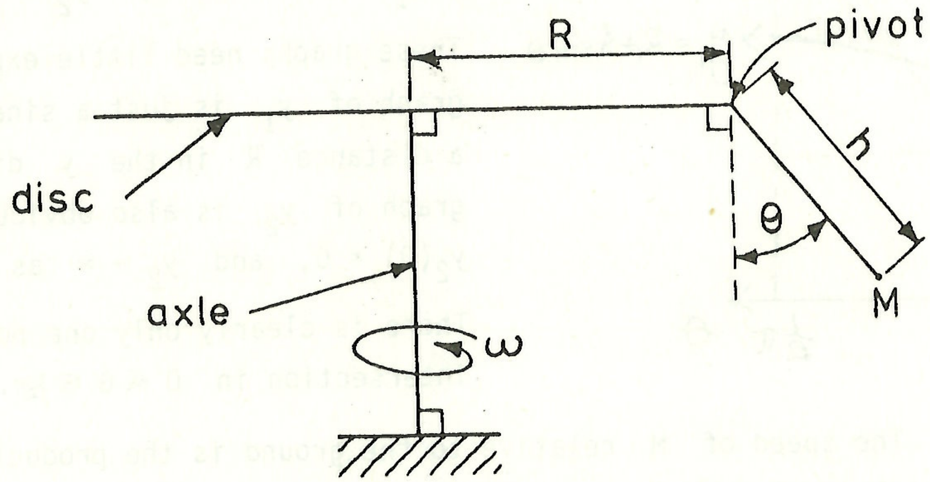
$$h\omega^2 = g. \tag{3}$$

Note that h depends only on ω, and not on θ or ℓ. Also, if t is the time taken for one revolution of the particle about the axis, then $\omega t = 2\pi$, so that $t = 2\pi/\omega = 2\pi(h/g)^{\frac{1}{2}}$.

We now turn to the 1982 H.S.C. problem

Problem 83.7.

A "chair-o-plane" at a fairground consists of seats hung from pivots attached to the rim of a horizontal circular disc, which is rotated by a motor driving its vertical axle. As the speed of rotation increases, the seats swing out. Represent a seat by a point mass M kg suspended by a weightless rod h metres below a pivot placed R metres from the axis of rotation. Assume that when the disc rotates about its axle with constant angular velocity ω radians per second, there is an equilibrium position in which the rod makes an angle θ with the vertical, as shown in the figure.



(a) Show that ω, θ satisfy the relation

$$(R + h \sin \theta)\omega^2 = g \tan \theta.$$

(b) Use a graphical means to show that, for given ω there is just one value of θ in the range $0 \leqslant \theta \leqslant \frac{\pi}{2}$ which satisfies this relation.

(c) Given $R = 6$, $h = 2$, $\theta = 60°$, and assuming that $g = 10 \text{ ms}^{-2}$, find the speed of M relative to the ground.

Solution: (a) Let T be the tension in the rod, then, as in (1) above,

$$T \cos \theta = M g.$$ Also, as in (2) above,

$$T \sin \theta = M\omega^2 (R + h \sin \theta),$$

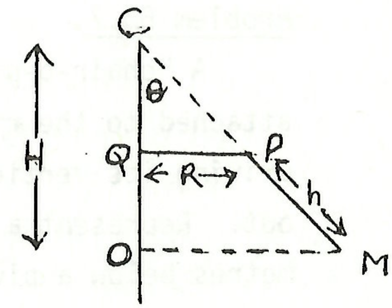
since $R + h \sin \theta$ is the radius of the path of the seat.

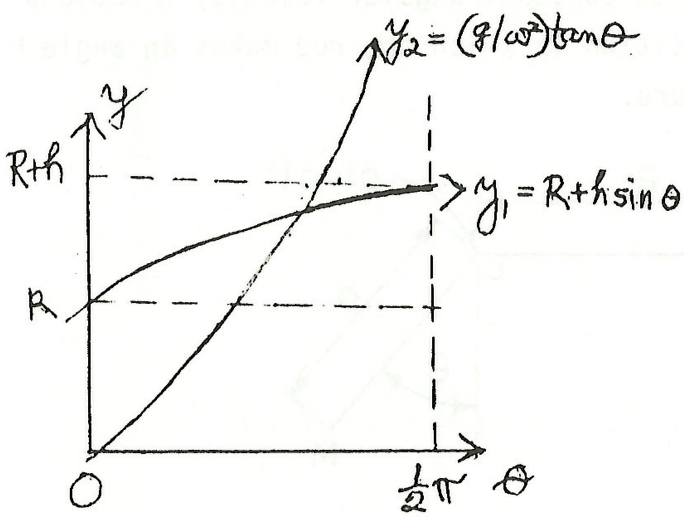
Thus $$(R + h \sin \theta)\omega^2 = T \sin \theta / M = g \tan \theta \qquad \text{(A)}$$

This result may also be established geometrically. Produce MP to meet the axis at C. Then, from (3) above $H = g/\omega^2$. But $H = CQ + QO = R\cot\theta + h\cos\theta$.

Hence $$R \cot \theta + h \cos \theta = g/\omega^2.$$

Multiply both sides by $\omega^2 \tan \theta$, and the result (A) follows.





(b) We draw, on the same diagram, the graphs, for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$

$$y_1 = R + h \sin \theta, \quad y_2 = \frac{g}{\omega^2} \tan \theta.$$

These graphs need little explanation. The graph of y_1 is just a sine curve, displaced a distance R in the y direction. The graph of y_2 is also obvious. Note that $y_2(0) = 0$, and $y_2 \to \infty$ as $\theta \to \frac{1}{2}\pi$.

There is clearly only one possible intersection in $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(c) The speed of M relative to the ground is the product of ω and the radius, which is $R + h \sin \theta$. Thus the speed v is given by

$$v = \omega(R + h \sin \theta), \text{ and } \omega = \left(\frac{g \tan \theta}{R + h \sin \theta}\right)^{\frac{1}{2}}.$$

When $\theta = 60^\circ$, $\sin \theta = \frac{1}{2}$, $\tan \theta = \sqrt{3}$, and for $R = 6$, $h = 2$, $R + h \sin \theta = 7$, $\therefore\ v = 7 \times 1.573 = 11\ \mathrm{ms}^{-1}$.