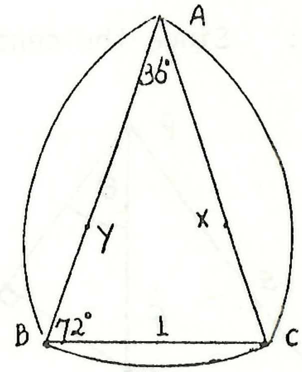
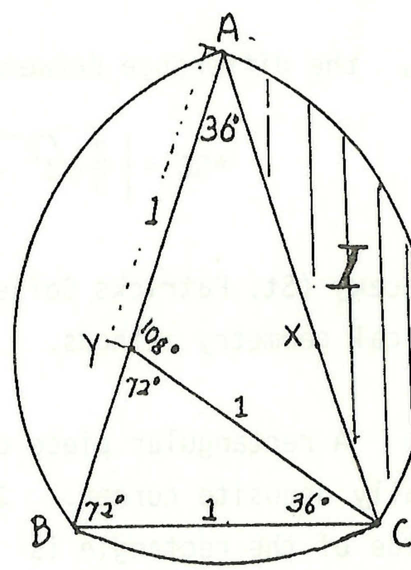
# SOLUTIONS TO PROBLEMS FROM VOLUME 19, NUMBER 2




Q. 563. In the figure, X and Y are the centres of the circular arcs AB and AC respectively, and A is the centre of the circular arc BC. The triangle ABC has the following dimensions:- $\angle A = 36°$, $\angle B = 72°$, length BC = 1. Find the area enclosed by the three circular arcs.

Solution: Gregory Low, (Year 12, Sydney Technical High School) supplies the accompanying figure, and writes




"Area I $= \frac{108°}{360°} \times \pi - \frac{1}{2}\sin 108°$

$$= \frac{3\pi}{10} - \frac{\sin 180°}{2}$$

In $\Delta ABC$ $\quad \frac{AB}{\sin 72°} = \frac{1}{\sin 36°}$

$$\therefore \quad AB = \frac{\sin 72°}{\sin 36°}$$

$$\therefore \quad \text{Area sector } ABC = \frac{36°}{360°} \times \pi \times \left(\frac{\sin 72°}{\sin 36°}\right)^2$$

$$= \frac{\pi}{10}\left(\frac{\sin 72°}{\sin 36°}\right)^2$$

Total Area = 2(Area I) + sector ABC

$$= \frac{3\pi}{5} - \sin 72° + \frac{\pi}{10}\left(\frac{\sin 72°}{\sin 36°}\right)^2 \text{ sq.u."}$$

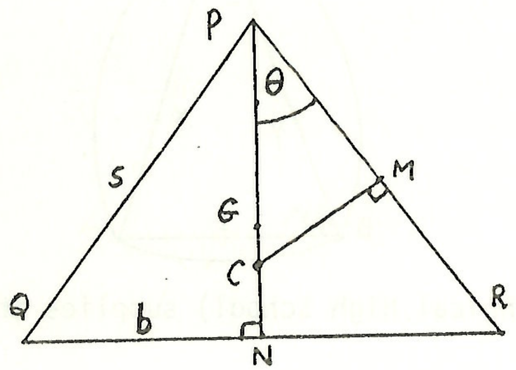
K. Boroczky (St. Patricks College, Strathfield) and J. Percival (Elderslie High School) take the calculation a little further, obtaining

$$\text{Area} = \frac{(15 + \sqrt{5})\pi - 5\sqrt{10 + 2\sqrt{5}}}{20} \approx 1.756.$$

Phuong van Nguyen (Fairfield High School) also supplied a correct solution involving Trigonometric functions.

Q. 564. In a triangle PQR, length PQ = length PR = s cms and length QR = 2b cms. Find the distance between the centroid and the circumcentre of the triangle.

Solution: Since the centroid G is known to lie at a point of trisection of the median PN, we have




$$*PN = \frac{2}{3}*PN$$
$$= \frac{2}{3}\sqrt{s^2 - b^2}\ .$$

The centroid C is the point of intersection of the side bisectors PN and MC. In $\Delta$ PMC *PM = *PC cos $\theta$,

whence $$*PC = \frac{1}{2}s \,/\, \frac{*PN}{*PR} = \frac{1}{2}\frac{s^2}{\sqrt{s^2 - b^2}}\ .$$

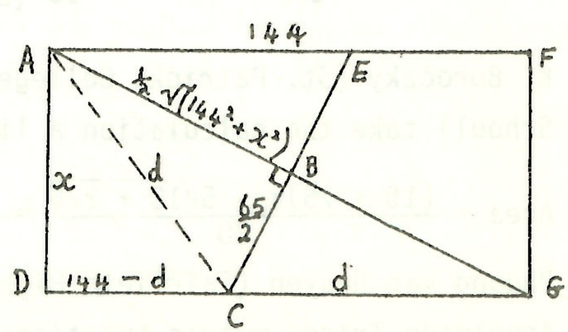
$\therefore$ *GC, the difference between *PG and *PC, is given by

$$*GC = \left|\frac{2}{3}\sqrt{s^2 - b^2} - \frac{1}{2}\frac{s^2}{\sqrt{s^2 - b^2}}\right| = \frac{|s^2 - 4b^2|}{6\sqrt{s^2 - b^2}}\ .$$

K. Boroczky (St. Patricks College, Strathfield) sent a correct working, using analytical geometry methods.

Q. 565. A rectangular piece of paper is folded to place one corner on the diagonally opposite corner. The length of the fold is 65 cms, and of the long side of the rectangle is 144 cms. Find the length of the short side of the rectangle.

Solution: Gregory Low, (Sydney Technical High School) supplies the figure in which CE is the fold when G is placed on A, and writes




"In $\Delta$ ABC $d^2 = \left(\frac{65}{2}\right)^2 + \frac{1}{4}(144^2 + x^2)$

$$d = \sqrt{(6240.25 + \frac{x^2}{4})}$$

*(continued over)*

In $\Delta$ ADC $\quad d^2 = x^2 + (144 - d)^2$

$$0 = x^2 + 144^2 - 288d$$

Substituting for d,

$$x^2 + 144^2 - 288\ (6240.25 + \frac{x^2}{4}) = 0$$

$$x^4 + 41472x^2 + 144^4 = 82944(6240.25 + \frac{x^2}{4})$$

$$x^4 + 20736x^2 - 87609600 = 0$$

Solving this quadratic in $x^2$ gives $x^2 = 3600$

$$x = 60 \text{ cm}$$

i.e. the shortest side is 60 cm".

Also neatly solved by J. Percival (Elderslie High School).

<u>Q. 566</u>. All the roots of the polynomial

$$x^5 - 10x^4 + ax^3 + bx^2 + cx - 32$$

are real and positive. Find a,b, and c.

<u>Solution:</u> The sum of the five roots is 10, and their product is 32. Hence their arithmetic mean, $\frac{10}{5}$, and their geometric mean, $\sqrt[5]{32}$, are equal. Using the theorem that the arithmetic mean of a set of positive numbers exceeds the geometric mean unless they are all equal, we deduce that all five roots must be equal to 2, and the polynomial is $(x - 2)^5$. Thus $a = 40$, $b = -80$, and $c = +80$.

John Percival (Elderslie High School) abd Gregory Low (Sydney Technical High School) both supplied the correct values for a,b and c without however giving the reason why these values are the only ones possible.

<u>Q. 567</u>. Show that every whole number, N, has some multiple which contains only the digits 0 and 9. e.g. if N = 571428, then 175 × N = 99999900.

<u>Solution:</u> Let $N = 2^{\alpha}5^{\beta}N'$ where $N'$ is an integer relatively prime to 10. If $\gamma = \max\{\alpha,\beta\}$ and $k = 2^{\gamma-\alpha}\ 5^{\gamma-\beta}$ then $kN = 10^{\gamma}N'$. It is then clear that

if we can show the existence of a natural number $c$ such that $cN'$ is one less than a power of 10, say $cN' = 10^{\delta} - 1$, then $ckN(= 10^{\gamma}cN')$ has a decimal representation consisting of $\delta$ nines followed by $\gamma$ zeros.

Since there are infinitely many different powers of 10, but only $N'$ different remainders when a number is divided by $N'$, it must be possible to find two powers of 10, say $10^s$ and $10^t$ $(t > s)$ which leave the same remainder. Thus $N'$ is a factor of $10^t - 10^s = 10^s(10^{\delta} - 1)$ where we have put $\delta = t - s$. However since $N'$ is relativity prime to $10^s$, $N'$ must be a factor of $10^{\delta} - 1$, as desired.

Q. 568. Without using calculus find the minimum value of

$$\frac{2 + 18x^4}{x^2}.$$

Solution: Since $\dfrac{2 + 18x^4}{x^2} = 2(\frac{1}{x} - 3x)^2 + 12$ and perfect squares are never negative, the values of the expression are all $\geqslant 12$, with equality when $3x = \frac{1}{x}$; viz. when $x = \pm \frac{\sqrt{3}}{3}$.

Correct solutions from G. Low (Sydney Technical High School) and J. Percival (Elderslie High School).

Q. 569. This is a long multiplication to calculate the square of a 4-digit number. All except the zero digits in the working have been obliterated, but the second last digit (i.e. the "tens" digit) of the answer is known to be odd. Find all the missing digits.

```
       X X X X
       X X X X
     X X X X X
   X X X X X
   X X X X
 X X X X
X O O X O X X X
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Solution: Since the first digit of the answer must be 1, the 4 digit number, N, lies between $\sqrt{10010111} \approx 3163.8$ and $\sqrt{10090999} \approx 3176.6$. The "tens" digit of $(10a + b)^2 = 100a^2 + 20ab + b^2$ is odd only if that is true for $b^2$; viz. $b = 4$ or $6$. This leaves only 4 possible values for N viz 3164, 3166, 3174 and 3176. Calculation shows that only the first of these gives a zero for the fifth digit of the answer. Thus the multiplication is 3164 × 3164 etc. Correct answer from G. Low (Sydney Technical High School).

Q. 570. "If two of my children are selected at random, likely as not they will be of the same sex", said the Sultan to the Caliph.
"What are the chances that both will be girls?" asked the Caliph.
"Equal to the chance that one child selected at random will be a boy" replied the Sultan.
How many children had the Sultan?

Solution: If the Sultan has b sons and g daughters, there are altogether $^{b+g}C_2 = \frac{(b+g)(b+g-1)}{2}$ different possible selections of 2 children, of which bg are "pigeon pairs". Thus

$$\frac{(b+g)(b+g-1)}{2} = 2\,bg \qquad (1)$$

The probability of selecting 2 girls is $\frac{g.(g-1)}{(b+g)(b+g-1)}$ and this is equal to $\frac{b}{b+g}$, the probability of choosing a boy in one random selection.

i.e. $$g(g-1) = b(b+g-1) \qquad (2)$$

It is not too difficult to solve (1) and (2) simultaneously to obtain $b = 6$, $g = 10$. The Sultan was blessed(?) with 16 children.

Q. 571. *(A puzzle which appeared in a Canadian Mathematical Journal, CRUX MATHEMATICORUM, December 1982.)*
The following quiz could be plenty of fun. Just remember to work in base 21.

```
NOW
 WE
ARE
SIX .. A summation of fact;
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That ONE divides SIX requires no tact;
That TWO is prime is perfectly valid;
But SIX must be perfect, and thus ends our ballad.

*(continued over)*

[i.e., replace the capital letters by whole numbers in the range 0, 1, 2, ..., 20 so that working in base 21, the addition sum is correct, and the other conditions are also satisfied. A perfect number is equal to the sum of its factors; in the usual decimal notation the first few are 6, 28, 496, 8128, 33550336.]

Solution: The only two perfect numbers with exactly 3 digits when using 21 as the number base are 496 (= $12\theta_{21}$ where $\theta = 13_{10}$) and 8128 (= $\alpha91_{21}$ where $\alpha = 18_{10}$). The former is too small to have a different 3-digit factor ONE, so we must have S = 18, I = 9, X = 1. Since O cannot be 1, we must have ONE $\geqslant 2 \times 21^2 = 882$. The only such factors of 8128 are 1016, 2032, and 4064 whose representations in base 21 are $268_{21}$, $4(12)(16)_{21}$ and $94(11)_{21}$ respectively. The third can be ruled out immediately since the digit 9 is already represented by I and cannot also be represented by O.

The first is also ruled out by the observation that if O = 2, N = 6 and E = 8 the first column of the addition sum forces W = 6, so that W and N are standing for the same digit.

This leaves as the only possibility O = 4, N = 12, E = 16 and to make the addition valid it is then easy to obtain W = 11, R = 13 and A = 5.

Finally $TWO_{21} = T \times 21^2 + 11 \times 21 + 4$ has 2 as a factor if T is odd and has factors 43, 53, 5, 13, and 11 for T = 6, 8, 10, 14, and 18 respectively. Thus T must be 2 (when TWO is the prime number $1117_{10}$).

Q. 572. A set consisting of integer numbers contains positive and negative numbers as well. If x and y are elements of the set then so are 2x and x + y. Prove that if any two elements of the set are chosen, their difference is also an element of the set.

Solution: Let the smallest positive number in the set, S, be m. Then clearly m + m, 2m + m. etc are all in the set; i.e. S contains all positive multiples of m. Let −x be any negative element of the set and let x = qm + r where r, the positive remainder after dividing x by m is a non negative integer less than m. But − x + (q + 1)m is then the sum of 2 elements of the set, i.e. m − r is in S. But m was the smallest positive member of S and $0 \leqslant r < m$. The only possibility is $r = 0$ and it follows that all negative elements of S are also exact multiples of m. If $-km$ is any one of them then, since $(k - 1)m$ is already known to be a member of the set $-km + (k - 1)m \in S$ i.e. S contains $-m$. Hence $-m + -m = -2m \in S$, and $-2m + -m = -3m \in S$; in fact all negative multiples of m are in S. Finally $m + -m = 0$ is in S. Thus S consists precisely of all multiples (positive, negative, or zero) of m and since the difference of 2 multiples of m is also a multiple of m, the required result is evident.

Q. 573. A convex polygon is dissected into triangles by non-intersecting diagonals. Every vertex of the polygon is a vertex of an odd number of such triangles. Prove that the number of edges of the polygon is a multiple of 3.

Solution: The triangles may be coloured using just two colours, $\alpha$, and $\beta$ such that triangles with a common edge have different colours. Of course this isn't possible with every figure dissected into triangles (e.g. Figure 1) but for polygons so dissected by drawing non intersecting diagonals the possibility is easy to prove by induction on the number of triangles:- [Choose any one of the triangles and colour it "$\alpha$" say. If it is now cut from the figure with scissors the remaining smaller pieces) can be 2-coloured by the induction hypothesis, and the colour $\beta$ can be used for the one triangle in each piece which has a scissor cut as one side.]

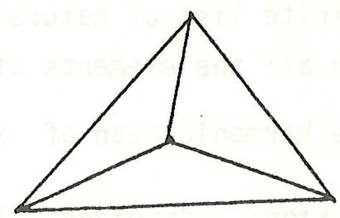

Fig 1.

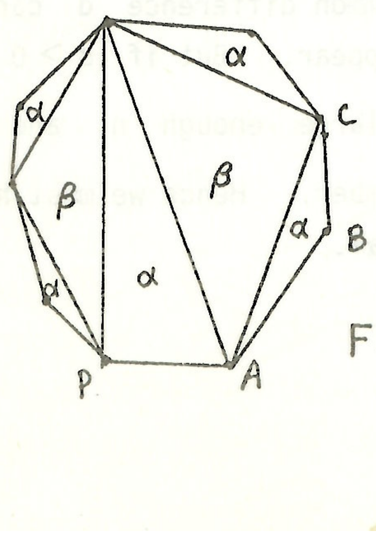


Fig 2

If in this colouring one triangle having an edge (or edges) of the polygon as (a) side(s), is given the colour "$\alpha$", (e.g. $\Delta$ ABC in Figure 2) then it is easy to see that all such triangles have colour "$\alpha$". Consider the vertex A for example. Since there are an odd number of triangles with vertex A, and the colours alternate from each to the next, we are back to colour "$\alpha$" when we reach the triangle with the next edge, AP. Hence all triangles with colour $\beta$ have sides consisting of just the diagonals of the polygon. The totality of sides of the "$\alpha$" triangles includes all the diagonals and in addition the sides of the polygon. Hence the number of sides of the polygon is the difference between the total number of sides of all the "$\alpha$" triangles and the total number of sides of all the "$\beta$" triangles. Since triangles have three sides, both these numbers are multiples of three and therefore the same is true of their difference. (Alternatively one could have used the fact that the number of diagonals to dissect an n-gon into triangles is $n - 3$, and it is also $3b$ where $b$ is the number of "$\beta$" triangles. Hence $n = 3 \times (b + 1)$.)

Q. 574. Prove that if every element, starting from the second one, of an infinite list of natural numbers is equal to the harmonic mean of its neighbours, then all the elements of the sequence are equal.

[The harmonic mean of $x$ and $y$ is $\frac{2xy}{x + y}$.]

Solution: As given, if of $x, w, y$, $w$ is the harmonic mean of $x$ and $y$, then $\frac{1}{w} = \frac{x + y}{2xy} = \frac{1}{2}(\frac{1}{y} + \frac{1}{x})$ and it follows that $\frac{1}{w} - \frac{1}{x} = \frac{1}{y} - \frac{1}{w}$. Let $a_1, a_2, a_3, \ldots, a_n, \ldots$ be the list of natural numbers. Then from the above observation

$$\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \ldots, \frac{1}{a_n}, \ldots$$

is a list of positive numbers with equal differences from each term to the next; i.e. an infinite arithemtic progression. The common difference $d$ cannot be negative, since eventually negative terms would appear. But if $d > 0$ then $\frac{1}{a_n} = \frac{1}{a_1} + (n - 1)d$ becomes greater than 1 for large enough $n$, and its recipiocal, $a_n$, cannot possibly be a natural number. Hence we must have $d = 0$, so that all terms of the sequence are equal.