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This issue we concentrate on some problems involving trigonometric functions and complex numbers.

Problem 84.5 (1983 common 3/4 unit paper).

Prove that

$$\frac{1 + \sin\theta - \cos\theta}{1 + \sin\theta + \cos\theta} = \tan(\theta/2)$$

Solution

Let $t = \tan\theta/2$, then $\cos\theta = (1-t^2)/(1+t^2)$, $\sin\theta = 2t/(1+t^2)$.
Then the left hand side of the identity is

$$\frac{1 + \frac{2t}{1+t^2} - \frac{1-t^2}{1+t^2}}{1 + \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}} = \frac{1 + t^2 + 2t - 1 + t^2}{1 + t^2 + 2t + 1 - t^2}$$

$$= \frac{2t + 2t^2}{2 + 2t} = \frac{t(2 + 2t)}{2 + 2t} = t.$$

as required.

Problem 84.6 (1983 common paper)

Find, to the nearest degree, all solutions of $3\cos 2A - \sin A + 2 = 0$, such that $0° < A < 360°$

Solution

We simply use the fact that $\cos 2A = 2\sin^2 A - 1$, and the equation reduces to

$$3(2\sin^2 A - 1) - \sin A + 2 = 6\sin^2 A - \sin A - 1$$
$$= (3\sin A + 1)(2\sin A - 1)$$
$$= 0$$

Hence $\sin A = \frac{1}{2}$ or $-\frac{1}{3}$, and the four solutions in the required range are 30°, 150°, 199°, 341°, correct to the nearest degree.

Problem 84.6 (1983 4 unit paper).

Given that $\sin x \sin y = \frac{1}{2}(\cos A - \cos B)$, find A, B in terms of x,y.

Hence prove that for any positive integer n

$$\sin x + \sin 3x + \sin 5x + \ldots \sin(2n-1)x = \frac{\sin^2 nx}{\sin x} \quad \ldots\ldots\ldots \quad (A)$$

Solution

The first part of the problem is a standard formula, since

$$\cos A - \cos B = 2\sin\frac{A+B}{2}\sin\frac{B-A}{2}$$

Thus $x = \frac{A+B}{2}$, $y = \frac{B-A}{2}$, and hence $A = x-y$, $B = x+y$, and, therefore

$$\sin x \sin y = \tfrac{1}{2}[\cos(x-y) - \cos(x+y)]$$

The second part may be proved by induction, as follows:

Assume that $S_k = \sin x + \sin 3x + \ldots + \sin(2k-1)x = \frac{\sin^2 kx}{\sin x}$

Then $S_{k+1} = \frac{\sin^2 kx}{\sin x} + \sin(2k+1)x = \frac{\sin^2 kx + \sin x \sin(2k+1)x}{\sin x}$

$$= \frac{\frac{1}{2}(1 - \cos 2kx) - \frac{1}{2}[\cos(2k+2)x - \cos 2kx]}{\sin x} = \frac{\frac{1}{2}[1 - \cos(2k+2)x]}{\sin x}$$

$$= \frac{\sin^2 (k+1)x}{\sin x}$$

Thus, if the result is true for k, it is also true for $k+1$. But for $k = 1$, $\frac{\sin^2 kx}{\sin x} = \sin x$, and the result is true.

The result for an arbitrary integer n follows by induction

There is however, a simple alternative proof

Cross multiply (A) by $\sin x$, and we need to find that

$$\sin x\, S_n = \sin x \sin x + \sin x \sin 3x + \ldots \sin x \sin (2n-1)x$$

$$= 1 - \tfrac{1}{2}\cos 2x + \tfrac{1}{2}(\cos 2x - \cos 4x) + \tfrac{1}{2}(\cos 4x - \cos 6x) + \ldots$$

$$+ \tfrac{1}{2}[\cos (2n-2)x - \cos 2nx]$$

using the result in the first part of this question. Now everything cancels except the very first and last terms on the right hand side, and so

$$\sin x\, S_n = 1 - \tfrac{1}{2}\cos 2nx = \sin^2 nx$$

and therefore $\qquad S_n = \sin^2 nx/\sin x$

This pretty cancelling technique can be used in other series for instance

Problem 84.7 Find the sum

$$S_n = 1.2.3 + 2.3.4 + 3.4.5 + \ldots (n(n+1)(n+2))$$

We just note that $k(k+1)(k+2)(k+3) - (k-1)k(k+1)(k+2)$

$$= k(k+1)(k+2)[k+3 - k+1] = 4k(k+1)(k+2)$$

For

$$k = 1, \quad 1.2.3 = \tfrac{1}{4}(1.2.3.4)$$
$$k = 2, \quad 2.3.4 = \tfrac{1}{4}(2.3.4.5 - 1.2.3.4)$$
$$k = 3, \quad 3.4.5 = \tfrac{1}{4}(3.4.5.6 - 2.3.4.5)$$

and so on, and then

$$k = n-1, \quad (n-1)n(n+1) = \tfrac{1}{4}[(n-1)n(n+1)(n+2) - n(n+1)(n+2)(n+3)]$$
$$k = n, \quad n(n+1)(n+2) = \tfrac{1}{4}[n(n+1)(n+2)(n+3) - (n-1)n(n+1)(n+2)]$$

Thus, when adding to find S_n, everything cancels out, and we find that

$$1.2.3 + 2.3.4 + 3.4.5 + 4.5.6 + \ldots n(n+1)(n+2) = \tfrac{1}{4}n(n+1)(n+2)(n+3)$$

Use this cancelling idea to sum the following two series

Problem 84.8

Express as partial fractions: $\frac{1}{k(k+1)}$; $\frac{1}{k(k+1)(k+2)}$.

Hence find the sums of the series

(a) $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \frac{1}{n(n+1)}$,

(b) $\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \cdots\cdots \frac{1}{n(n+1)(n+2)}$.

Solutions in the next issue of Parabola.