PROBLEM SECTION

Q. 648. Prove that no perfect square except 0 is the product of 6 consecutive integers.

Q. 649. O is a point inside a convex n-gon $P_1P_2 \ldots P_n$ of area A such that $OP_1^2 + OP_2^2 + \ldots + OP_n^2 = 2A$. Prove that the n-gon is a square, and O is the intersection of the diagonals.

Q. 650. The bisectors of the angles C and D of a convex quadrilateral ABCD meet at a point P on AB such that $C\hat{P}D = D\hat{A}B$. Prove that P is the mid point of AB.

Q. 651. Given distances $a_1, a_2, \ldots, a_n$ satisfying

$$a_i < \tfrac{1}{2}(a_1 + a_2 + \ldots + a_n) \quad \text{for} \quad i = 1, 2, \ldots, n$$

prove that there exists a closed plane polygon with these sides (in the given order).

Q. 652. N is a 4 digit number, and M is obtained from N by interchanging the first pair of digits with the last pair. (e.g. if N = 9726, M = 2697). Find N given that $N^2 - M^2$ is a perfect square.

Q. 653. The sum of two positive integers is 10000000000. The digits of the two numbers are the same but not in the same order. Prove that both numbers are divisible by 5.

Q. 654. A class of n students sit for 2 examination papers, the marks being added to ive the final result. An entry (4,7,6) opposite a name in the roll book indicates that the student in question came fourth in the first paper, seventh in the second, and sixth in the whole examination after the marks were added. Note that (1,1,2) is an impossible entry, and so is (16,17,5) if

$n = 20$. Find conditions on a,b,c in order that (a,b,c) is an impossible result.

Q. 655. Given an acute angled triangle ABC, show how to construct a point P such that the circumcircles of the triangles ABC, ABP, BCP and CAP all have equal radii (but do not coincide; i.e. P does not lie on the circumcircle of ΔABC). Prove your construction. Discuss the case when ΔABC is not acute angled.

Q. 656. Let E be a set containing n elements and let $A_1, A_2, \ldots, A_n$ be distinct non-empty subsets of E such that

$$A_i \cap A_j \neq \phi, \quad i \neq j \tag{1}$$

a) Show that $r \leq 2^{n-1}$.

b) If B is an arbitrary subset of E, show that either $A_i \cap B \neq \phi$ for $i = 1,2, \ldots, r$ or $A_i \cap \bar{B} = \phi$ for $i = 1,2, \ldots, r$, where $\bar{B}$ is the complement of B in E.

c) Show that if $r < 2^{n-1}$ any given collection of subsets $A_1, \ldots, A_r$ satisfying (1) can be extended by finding subsets $A_{r+1}, \ldots, A_{2^n-1}$, so that the collection of $2^n - 1$ subsets still has property (1).

[ϕ denotes the empty set.]

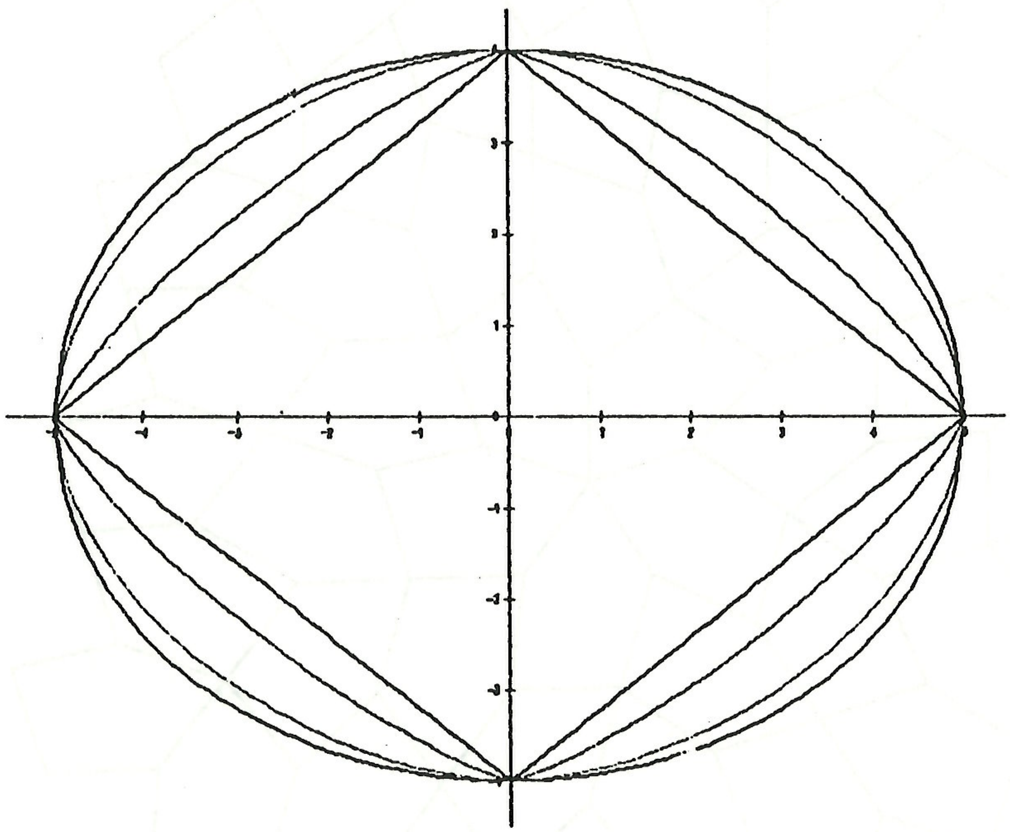
Q. 657. We have 30 locked boxes and 30 keys, each opening just one lock. Each box has an opening through which we throw one key at random.

i) We break one of the boxes. What is the probability that we can open all the other boxes without breaking them?

ii) If we initially break two boxes instead of one, what is the probability that we can open all the other boxes without breaking them?

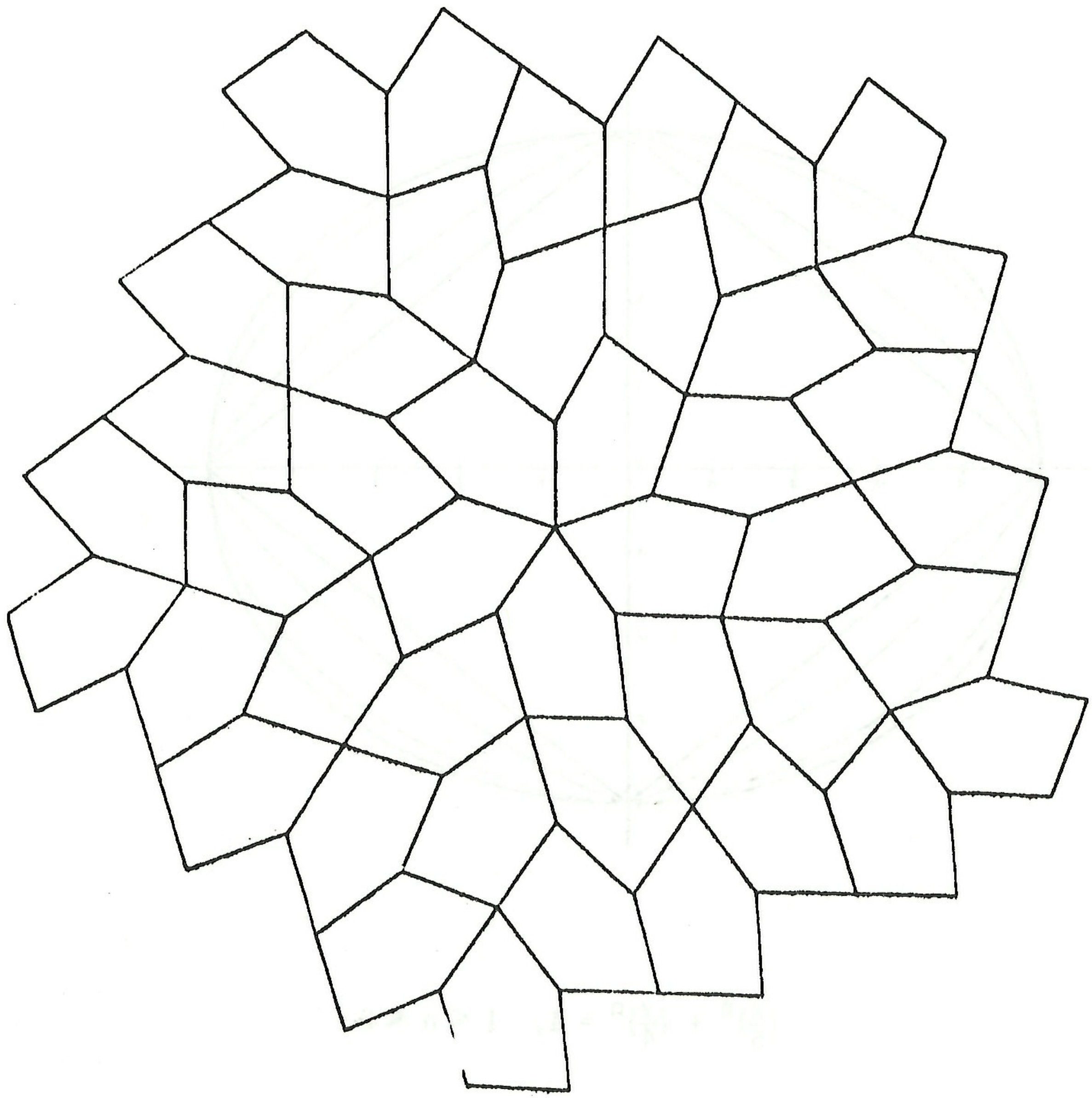
Q. 658. The triangle ABC is isosceles and has a right angle at B. The point M is on the circumcircle of ABC. Find all positions of M such that it is possible to construct a triangle from the segments BC, MA, MC.

Q. 659. In chess a king can move to any adjacent unoccupied square, horizontally, vertically, or diagonally. What is the maximum number of kings that can be placed on an 8×8 chessboard so that each king has at least one square to which it can move. (For example, on a 2×2 board, the answer would be three.) Justify your answer.



$$\left(\frac{x}{5}\right)^n + \left(\frac{y}{4}\right)^n = 1, \quad 1 \leq n \leq 2.$$

MERRY CHRISTMAS



HIRSCHHORN'S PENTAGONAL TILING