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CHANGING WATER TO WINE

In a recent edition of Mathematical Spectrum (the English equivalent of Parabola) the following famous problem was discussed:

Suppose you have two containers, one of which holds exactly 5 litres and one of which holds exactly 3 litres. How would you obtain 4 litres of water using only these containers?

The solution of this question may be shown using ordered pairs of numbers to describe the contents of the two containers, for example (5,0) means that the 5-litre container is full and the 3-litre container is empty; (0,3) means that the 5-litre conainer is empty and the 3-litre container is full; also

$$(5,0) \rightarrow (2,3)$$

means fill the empty 3-litre container from the full 5-litre container, leaving 2 litres left in the 5-litre container. Remembering that the only operations allowed are to empty a container (in the sink or other container) or fill a container (from a tap or the other container), a solution to the above problem is:

$$(0,0) \rightarrow (5,0) \rightarrow (2,3) \rightarrow (2,0) \rightarrow (0.2) \rightarrow (5,2) \rightarrow (4,3)$$

This problem suggests the following generalization:

What volumes of water can we get using two containers which will hold N litres and n litres?

To start answering this question, we will suppose $N \geq n$ and note that at each step we can only transfer water from one container to the other, or obtain or throw away n litres or N litres. This leads to:

Theorem 1 Every volume of water we may obtain must be able to be written in the form $Nx + ny$ where x, y are integers.

Thus, for example, we cannot get any fractions of a litre in our answers.

Now, if d divides both N and n, then it is easy to see that d divides the number $Nx + ny$ for every pair of integers x, y and so

Theorem 2 If d is a common divisor of N and n, then we cannot get any volume of water which is not divisible by d.

Thus if we have a 6-litre container and a 4-litre container we cannot get an odd number of litres. We can however get 6 litres, 4 litres (both obvious) and 2 litres by $(6, 0) \to (2, 4)$.

Because of theorem 2, we will suppose from now on that the numbers N, n have no common divisors (except 1 of course) and see if we can get any volume of water. We first note that for any integers, the sequence

$$(s, 0) \to (s - n, n) \to (s - n, 0) \to (s - 2n, n) \to (s - 2n, 0) \to \ldots$$

consists of getting the <u>remainder when</u> s <u>is divided by</u> n and we will call this number <u>mod (s, n)</u>. For example, mod (5, 3) = 2, mod (23, 5) = 3. Also, if $s > n$, then the sequence

$$(0, s) \to (N, s) \to (N + s - n, n) \to (N + s - n, 0) \to (N + s - 2n, n)$$
$$\to (N + s - 2n, 0) \to \ldots$$

shows that

Theorem 3 If we can get s litres of water, then we can get mod (s, n) and mod (s + N, n) litres of water.

We are now in a position to prove our main result:

Theorem 4 If we have two containers holding N litres and n litres of water where the numbers N, n have no common divisors except 1, then we can get any number of litres up to N.

Proof - Since we can obviously get 0 litres, theorem 3 tells us that we can get mod (N, n) litres and then get

$$\text{mod } (N + N, n) = \text{mod } (2N, n) \text{ litres,}$$
$$\text{mod } (2N + N, n) = \text{mod } (3N, n) \text{ litres etc.}$$

Now there is a useful result in the theory of numbers which says that the sets

$$\{ \text{mod } (N,n), \text{mod } (2N, n), \ldots, \text{mod } ((n - 1) N, n)\}$$
$$\text{and} \quad \{ 1, 2, \ldots n - 1\}$$

are the same (although usually in a different order). This means that we can get any number of litres up to n litres, and the intermediate steps show that we can get s litres for any s (up to N of course)

<u>Example</u> The following sequence shows how to get 6 litres using a 7-litre and a 3-litre container:

$$(7, 0) \to (4, 3) \to (4, 0) \to (1, 3) \to (1, 0) \to (0, 1) \to (7, 1) \to (5, 3)$$
$$\to (5, 0) \to (2, 3) \to (2, 0) \to (0, 2) \to (7, 2) \to (6, 3).$$

Notice that this does not necessarily yield the shortest way of getting any number of litres of water, since we could also have done

$$(7, 0) \to (4, 3) \to (0, 3) \to (3, 0) \to (3, 3) \to (6, 0)$$

which is much shorter.

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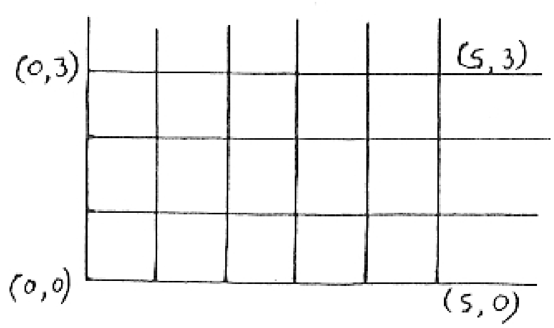
Having solved our problem of getting various amounts of water, now let us suppose that someone presents you with a bottle containing 8 litres of wine and asks you to share out 4 litres using our original 5-litre and 3-litre containers. The difference between this and the earlier ques tion is tht we only have 8 litres of wine with which to fill our containers, and of course you would not simply empty a 3-litre container of wine down the sink! Thus the operation (2,3) → (2,0) in our original solution is wasteful.

Despite what has just been said, a moment's thought will show that the above operation may be interpreted as pouring the contents of the 3-litre container back into the original bottle (instead of down the sink). This can be represented by using ordered triples (instead of pairs) to describe the contents of the 8-litre bottle and our 5-litre and 3-litre containers, where the sum of three numbers of the ordered triple is always the same. Using this notation, the "wine" solution to our original problem becomes

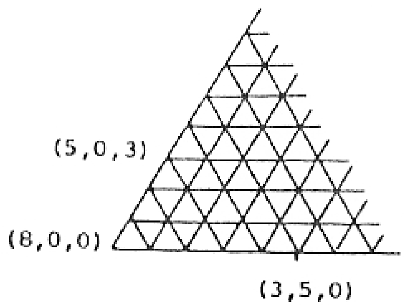
(8, 0, 0) → (3, 5, 0) → (3, 2, 3) → (6, 2, 0) → (6, 0, 2) → (1, 5, 2) → (1, 4, 3)

Similarly the results of theorems 3 and 4 are still true in this context provided our bottle of wine originally has at least $N + n$ litres of wine. We are still (even in this case) left with the question of the most efficient way of getting the amount of wine we want.

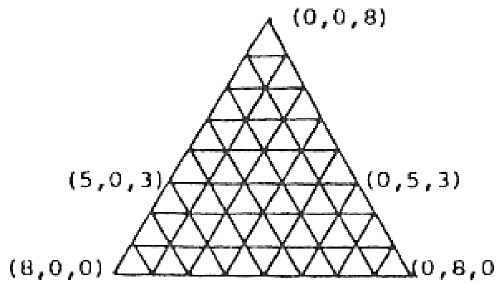
For a way of solving the harder question we turn to the use of graphs. We are used to representing ordered pairs of numbers as points on a plane, so that the contents of our 5-litre and 3-litre containers can be represented by the intersection of any of the following lines:



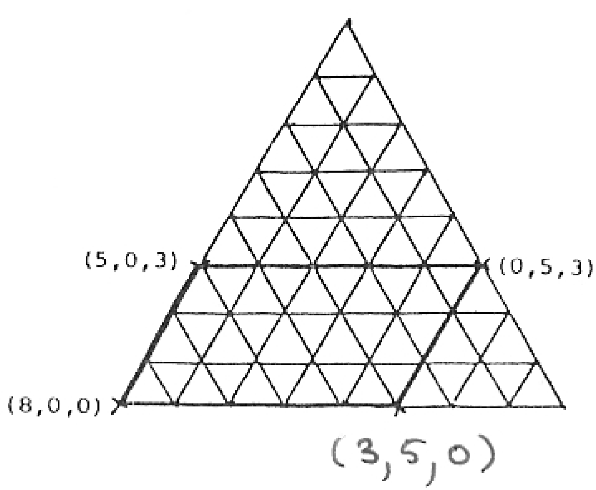
Our wine version of this diagram is to replace this rectangular grid by a triangular grid (Note that we do not use 3-dimensional space as the contents of the bottle depend on the contents of the 2 containers and so is not a "third dimension"). In this triangle, we can think of the sides of the triangle as our axes, with the base corresponding to when the 3-litre container is empty and the left-hand side corresponding to when the 5-litre container is empty.



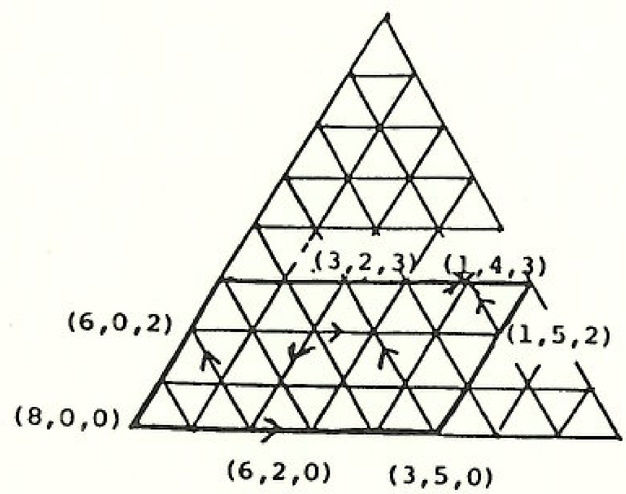
To be fair to the 8-litre bottle, we will treat it as an 8-litre container and use the third side of the triangle for the case when it is empty. Finally, since none of our containers can hold more than 8 litres, we will use an equilateral triangle with sides of length 8 units. Thus the contents of our three containers can be thought of as some of the points in the triangle:



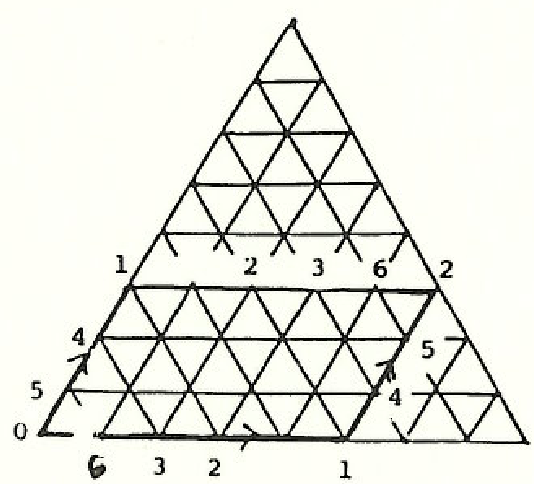
For example, the point marked (8,0,0) represents the fact that the 8-litre container is full and the other two are empty, and the point marked (0,5,3) represents the fact that the 8-litre container is empty and the other two are full. Similarly the point marked (0,8,0) represents the fact that the 5-litre container has 8 litres and the other two are empty - which is clearly impossible. For this reason we have to restrict ourselves to the parallelogram joining the points (8,0,0), (3,5,0), (0,5,3) and (5,0,3):



To use this triangle, we note that a line parallel to the base represents emptying the 8-litre container into the 5-litre container, a line parallel to the left hand side represents emptying the 8-litre container into the 3-litre container and a line parallel to the right hand side represents emptying the 5-litre container into the 3-litre conainer. So our original solution can now be represented by:

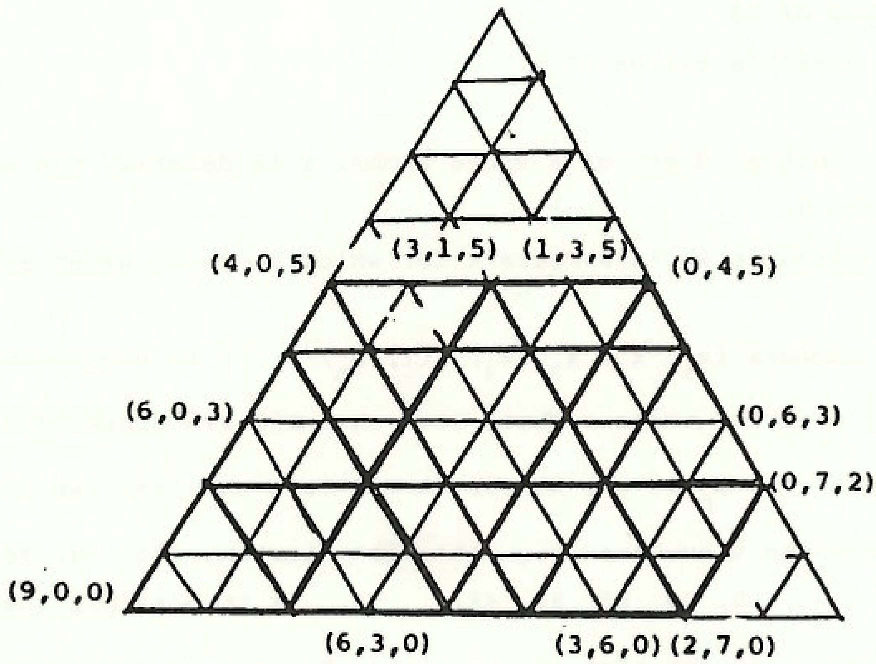


In fact, we can also use the triangle to find the quickest solution to a problem. Since we are starting with all the wine in the 8-litre conainer, we label the vertex (8, 0, 0) with the number 0 to indicate that we require no steps to get there. Now label all vertices of the parallelogram which we can reach immediately with the number 1 (in our example, these will be (3, 5, 0) and (5, 0, 3)). We then label any point we can reach immediately from any 1-point with the number 2 etc. For example, when we first reach a point with 4 as one of its co-ordinates, our triangle looks like:



and so the given solution to our original problem is actually the quickest.

As another example, imagine the problem of getting 3 litres of wine from a bottle of 9 litres using a 7-litre container and a 5-litre container (you are showing it with 2 friends). The following diagram shows all the possible paths starting at the vertex (9, 0, 0) and since none of them passes through any point with a co-ordinate equal to 3, we can see that the problem is impossible.



You might like to experiment with other combinations and decide which ones are possible and which are impossible. In particular, you might like to try to show that if N, m, n are three numbers with no common factor, where $N > m > n$ and

$$m + n - 2 \le N \le 2m + 2n + 1$$

then we can get any number of litres of wine (up to N of course) from a bottle of N litres of wine, an m-litre container and an n-litre container.