Parabola Volume 24, Issue 3 (1988)

FINITE GEOMETRY AND ... TOMATO PLANTS?

by

BOB HART

Hawker College

We are all familiar with the co-ordinate geometry based on the real axes and ordered pairs (x, y) where x and y may be any element of the infinite set of real numbers.

Suppose we only allow x and y to take values from a finite set?

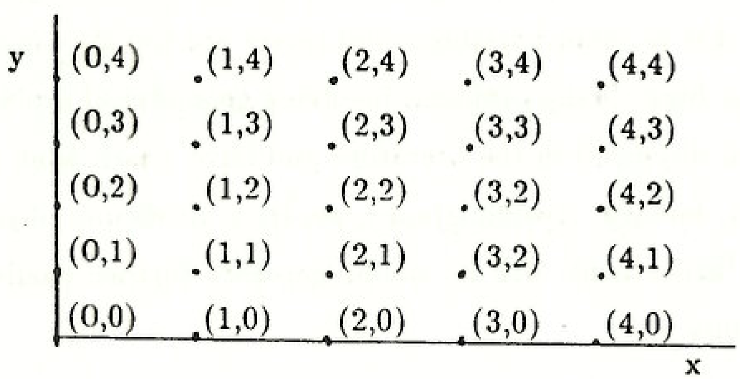
Consider, for example, the finite arithmetic modulo 5:

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

x	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

We call the elements $0, 1, 2, 3, 4$ and the arithmetic defined by the above tables I_5.

By setting up a rectangular co-ordinate system and plotting the ordered pairs $(0,0)$ $(0,1)$..........$(4,4)$ we get the pattern in the following diagram:



We define a <u>point</u> to be an ordered pair (x, y) where x and y are in I_5 and a <u>line</u> to be a set of ordered pairs satisfying an equation of the form $ax + by = c$ where a, b, c are in I_5 and a, b are not <u>both</u> zero.

Note: we need to be careful because some equations which look different may in fact represent the same set of points. For example, here are 3 equations satisfied by the same ordered pairs:

$$x + 4y = 2 \qquad (1)$$

$$2x + 3y = 4 \qquad (2)$$

$$4x + y = 3 \qquad (3)$$

Equation (2) is equation (1) multiplied by 2 according to the multiplication table for I_5.

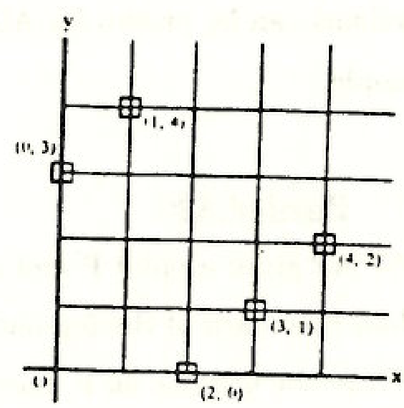
Let us now plot the points of the line $2x + 3y = 4$:

This is equation (2) above, and by multiplying by 2 we get equation (3) or $4x + y = 3$.

i.e. $y = -4x + 3$

Since $4 + 1 = 0$, then $-4 = 1$.

The 5 points of the line $2x + 3y = 4$.



Hence the equation becomes $y = x + 3$

By completing the table below we get the required points.

x:	0	1	2	3	4
y:	3		0		2

The points are plotted in the above diagram.

We will call this system of analytical geometry AG5

Clearly this geometry has 25 points, but how many lines?

Solution:

There are 5 lines of the form $x = c$ (5 choices of c). All other lines are of the form $y = mx + r$. There are 5 possible choices each for m and r, and each choice gives a new lines. Hence there are $5 \times 5 = 25$ lines of this form, and so there are 30 lines altogether.

How many lines lie on each point?

Solution:

Choose a point, say $P(x_1, y_1)$. One line on P is of the form $x = c$ (It is $x = x_1$). The remaining lines on P are of the form $y = mx + r$. We must have $y_1 = mx_1 + r$ and each of the 5 values of m determines uniquely a value for r.

Hence there are 5 such lines.

Therefore, there are 6 lines on each point.

Exercise: Prove that there are 5 points on each line.

The geometry AG5 satisfies the following axioms:

A1: Given 2 distinct points P and Q there is exactly one line that contains them both.

A2: Given a line l and a point P not on l, there is exactly one line through P not containing any of the points of l.

A3: There are at least 4 points, no 3 of which are collinear.

These axioms can be proven for AG5 using the information we have already established. For example,

Proof of A2:

We are given a point P and a line l. We have shown that P must have 6 lines on it. Join P to each of the 5 points on l and we get 5 of the 6 lines on P. Hence the 6th line on P must be the unique line not containing any of the points of l.

(Note that A2 is equivalent to the parallel axiom of Euclidean Geometry; it is also known as Playfair's Axiom).

Now to The Tomato Plant Problem

Suppose we wish to test 5 different water dosages and 5 different fertilizers in all possible combinations in our garden. The tomato plants must be arranged in a square array so that all will get much the same wind and sunlight and will be in soil of similar quality. So that the test is as random as possible, each water dosage and each fertilizer must occur exactly once in each row and column of the square garden.
How can we easily obtain such a combination?

Let the 5 water dosages be represented by a,b,c,d,e and let the 5 fertilizers be represented by A,B,C,D,E. The 5 water dosages can be arranged so that each occurs once in each row and column as shown in the diagram.
Such an array is called a LATIN SQUARE.

e	d	c	b	a
d	c	b	a	e
c	b	a	e	d
b	a	e	d	c
a	e	d	c	b

Our problem now is to find a Latin Square for the 5 fertilizers so that if we superimpose it on the one given for the water dosages then each of the 25 combinations Aa, Ab, ...Ee occurs exactly once. Two such Latin Squares are said to be orthogonal.

Solution:

In our geometry AG5, consider the points on the 5 lines $y = x + r, r$ in I_5. When $r = 0$ we get the 5 points (0,0) (1,1) (2,2) (3,3) and (4,4). Label these points "a". When $r = 1$ we get the 5 points (0,1) (1,2) (2,3) (3,4) and (4,0). Label these points "b".

Continue in this way - when $r = 2$ label the points "c"
when $r = 3$ label the points "d"
when $r = 4$ label the points "e"

We now have labelled all 25 points of AG5 and the result is the Latin Square given for the water dosages. Now, to get an orthogonal Latin Square we consider the points on the 5 lines $y = 2x + r, r$ in I_5. When $r = 0$ we get the 5 points (0,0) (1,2) (2,4) (3,1) and (4,3). Label these points "A". When $r = 1$ we get the 5 points (0,1) (1,3) (2,0) (3,2) and (4,4). Label these points "B".

Continue in this way - when $r = 2$ label the points "C"

when $r = 3$ label the points "D"

when $r = 4$ label the points "E"

Again, all 25 points of AG5 have been labelled, giving us 2 Latin Squares and behold! - they are orthogonal:

eE	dC	cA	bD	aB
dD	cB	bE	aC	eA
cC	bA	aD	eB	dE
bB	aE	eC	dA	cD
aA	eD	dB	cE	bC

Hence we have a solution to the tomato plant problem. We can get a pair of orthogonal Latin Squares by using any pair of lines in AG5 $y = m_1x+r,\ y = m_2x+r$; we only need to choose m_1 and m_2 to be non-zero and not equal.

Further, it turns out that we can obtain orthogonal Latin Squares using this method in a geometry AGn as long as n is the power of a prime number.

What about other forms of n, such as $n = 6$? In 1782 Leonard Euler conjectured that there are no 2 orthogonal Latin Squares of order 6. This was finally proven in 1900 by G. Tarry and he proved it by brute force; i.e. he listed all possible combinations and showed no 2 of the squares were orthogonal. Euler had also conjectured that there were no 2 orthogonal Latin squares for any integer $n = 4k + 2$. In 1959 it was proven that there are 2 such squares for all $n = 4k + 2$ except 6. Sadly, Euler was way off!!

Finally, here is an orthogonal pair of squares for $n = 10$:

aA	eH	bI	hG	cJ	jD	iF	dE	gB	fC
iG	bB	fH	cI	hA	dJ	jE	eF	aC	gD
jf	iA	cC	gH	dI	hB	eJ	fG	bD	aE
fJ	jG	iB	dD	aH	cI	hC	gA	cE	bF
hD	gJ	jA	iC	eE	bH	fI	aB	dF	cG
gI	hE	aJ	jB	iD	fF	cH	bC	eG	dA
dH	aI	hF	bJ	jC	iE	gG	cD	fA	eB
bE	cF	dG	eA	fB	gC	aD	hH	iI	jJ
cB	dC	eD	fE	gF	aG	vA	iJ	jH	hI
eC	fD	gE	aF	bG	cA	dB	jI	hJ	iH

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