SOLUTIONS OF PROBLEMS 762-771

Q.762 Simplify the sum

$$\sec\theta + \sec\theta\sec 2\theta + \sec 2\theta\sec 3\theta + \cdots + \sec(n-1)\theta\sec n\theta$$

$$= \sum_{k=1}^{n} \sec(k-1)\theta\sec k\theta$$

ANSWER $\frac{1}{\sin\theta}(\tan n\theta - \tan(n-1)\theta) = \frac{\sin n\theta\cos(n-1)\theta - \cos n\theta\sin(n-1)\theta}{\sin\theta\cos n\theta\cos(n-1)\theta}$

$= \frac{\sin(n\theta-(n-1)\theta)}{\sin\theta\cos n\theta\cos(n-1)\theta} = \sec n\theta\sec(n-1)\theta$ for $n = 1, 2, \cdots$

$\therefore \sec 0\theta\sec 1\theta + \sec\theta\sec 2\theta + \cdots + \sec(n-1)\theta\sec n\theta$

$= \frac{1}{\sin\theta}(\tan\theta - 0) + \frac{1}{\sin\theta}(\tan 2\theta - \tan\theta) + \cdots + \frac{1}{\sin\theta}(\tan n\theta - \tan(n-1)\theta)$

$= \frac{\tan n\theta}{\sin\theta}.$

(If $\sin\theta = 0$ every term in the sum is equal to +1 if $\cos\theta = 1$, and to -1 if $\cos\theta = -1$, and the sum is n or $-n$. If θ is such that $\tan k\theta$ is undefined for some $k \leq n$, then the kth term $\sec(k-1)\theta\sec k\theta$ is also undefined. For all other values of θ our working applies).

Q.763 Take the usual decimal representation of any whole number, x. If the number of digits is odd, place a zero at the beginning (e.g. replace 2350002 by 02350002). Break it up into consecutive pairs of digits, and let $T(x)$ be the sum of all these two-digit whole numbers.

e.g. $T(2350002) = 02 + 35 + 00 + 02 = 39.$

(i) Show that for any $x, x - T(x)$ is a multiple of 99, and deduce that $T(x)$ and x leave the same remainders on division by 9, and also on division by 11

(ii) If $x = 1989^{1989}$, show that $T(x) < 400000$ and calculate $T(T(T(x)))$ exactly.

ANSWER (i)

$$10^{2n} - 1 = \overbrace{99\cdots 9}^{2n \text{ digits}} = 99 \times \overbrace{10101\cdots 01}^{n1's} \qquad (*)$$

If x is represented in decimal notation by the string $a_{2n+1}a_{2n}a_{2n-1}a_{2n-2}\cdots a_1a_0$, (so $x = a_1a_0 + a_3a_2 10^2 + a_5a_4 10^4 + \cdots + a_{2n+1}a_{2n}10^{2n}$), then

$$x - T(x) = a_1a_0(10^0 - 1) + a_3a_2(10^2 - 1) + \cdots + a_{2n+1}a_{2n}(10^{2n} - 1)$$

Since, from (*), 99 is a factor of every term on the R.H.S., it is a factor of $x - T(x)$.

$\therefore$ Let $x = T(x) + 99k$. If $T(x) = 9q + r$ where $0 \le r < 9$ then $x = 9(q + 11k) + r$, so x and $T(x)$ leave the same remainder on division by 9. A similar argument applies for division by 11.

(ii) $x = 1989^{1989} < 10000^{2000} = 10^{8000}$, so x has at most 8000 digits.

$T(x)$ is the sum of 4000 (at most) 2 digit numbers.

$\therefore T(x) < 4000 \times 100 = 400000$

$\therefore T(T(x))$ is the sum of at most 3 two digit numbers.

$$T(T(x)) < 300. \ \therefore 0 < T(T(T(x))) \le 2 + 99.$$

Since 1989 is divisible by 9, so is x, and, because of (i), so is $T(x), T(T(x))$ and $T(T(T(x)))$.

We shall see that x, and hence each of $T(x)$, $T(T(x))$, and $T(T(T(x)))$, leaves a remainder of 5 on division by 11. It follows that $T(T(T(x)))$ must be equal to

27, the only multiple of 9 less than 101 which leaves the remainder 5 on division by 11.

(Proof that $1989^{1989} = 11k + 5$ for some integer k.

Since $(11q_1 + r_1)(11q_2 + r_2) = 11(11q_1q_2 + q_1r_2 + q_2r_1) + r_1r_2$ it follows that if a and b leave remainders r_1 and r_2 on division by 11, the product ab leaves the same remainder as r_1r_2. Using this, since $1989 = 11 \times 180 + 9$,

1989^2 leaves the same remainder as 9^2, namely 4;

$1989^3 = 1989^2 \times 1989$ leaves the same remainder as 4×9, namely 3;

$1989^4 = 1989^3 \times 1989$ leaves the same remainder as 3×9, namely 5;

$1989^5 = 1989^4 \times 1989$ leaves the same remainder as 5×9, namely 1.

$1989^{m+5} = 1989^m \times 1989^5$ leaves the same remainder as $1989^m \times 1$.

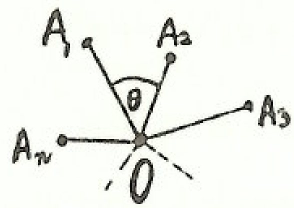
$\therefore$ 1989^{5q+r} leaves the same remainder as 1989^r

In particular, since $1989 = 5 \times 397 + 4$

1989^{1989} leaves the same remainder as $(1989)^4$, namely 5. Q.E.D.)

Q.764 We have n points in the plane, no two distances between points being equal. A straight line interval is drawn connecting each of the points to the nearest of the others. Prove that no point lies on more than 5 of the intervals.

ANSWER Suppose on the contrary that the point O is joined to points $A_1, A_2, \cdots, A_n$ where $n \geq 6$. We may assume without loss of generality that $A_1\hat{O}A_2 = \theta$ is the smallest of the n angles at O. Then $\theta \leq 60°$. In ΔA_1OA_2 at least one of $O\hat{A}_1A_2$ and $O\hat{A}_2A_1$ must exceed $60°$. (They cannot both equal $60°$ since the lengths $0A_1$ and $0A_2$ are not equal).



For definiteness suppose $O\hat{A}_2A_1 > O\hat{A}_1A_2$ and $O\hat{A}_2A_1 > 60°$. Then OA_1 is longer than OA_2 and longer than A_1A_2 (since it is opposite the largest angle in the triangle). But the line interval OA_1 is drawn either because O is the closest point to A_1, or A_1 is the closest point to O. Thus we have arrived at a contradiction, which establishes the desired conclusion.

Q.765 If n is any whole number, let $d(n)$ denote the number of different factors of n, and let $\sigma(n)$ denote their sum. For example, the factors of 12 are 1,2,3,4,6 and 12, so $d(12) = 6$ and $\sigma(12) = 1 + 2 + 3 + 4 + 6 + 12 = 28$. If the factorisation of n into prime numbers is $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$

show that $d(n) = (1 + a_1)(1 + a_2) \cdots (1 + a_k)$.

and $\sigma(n) = \left(\frac{p_1^{a_1+1}-1}{p_1-1}\right)\left(\frac{p_2^{a_2+1}-1}{p_2-1}\right)\cdots\left(\frac{p_k^{a_k+1}-1}{p_k-1}\right)$

[Since $12 = 2^2 \times 3$, these formulas give $d(12) = (2 + 1) \times (1 + 1) = 6$

and $\sigma(12) = \left(\frac{2^3-1}{2-1}\right)\left(\frac{3^2-1}{3-1}\right) = 7 \times 4 = 28$.]

ANSWER m is a divisor of n iff $m = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ where no exponent e_j exceeds the corresponding exponent a_j in the factorisation of n.

Thus e_j is one of $\{0, 1, 2, \cdots, a_j\}$. There are $(a_j + 1)$ possible values for $e_j, j = 1, 2, \cdots k$; and it follows that there are $(a_1+1)(a_2+1)\cdots(a_k+1)$ different divisors of n.

For the second result, note that when the product

$$(1+p_1+p_1^2+\cdots+p_1^{a_1})(1+p_2+p_2^2+\cdots+p_2^{a_2})\cdots(1+p_k+\cdots+p_k^{a_k}) \qquad (1)$$

is expanded out, every term in the resulting expression is of the form $p_1^{e_1}\cdots p_j^{e_j}\cdots p_k^{e_k}$ where $0 \le e_j \le a_j$. Hence every term is a factor of n, and conversely every factor of n occurs exactly once, so that the product (1) is equal to $\sigma(n)$.

Since $(1+p_j+\cdots+p_j^{a_j}) = \dfrac{p_j^{a_j+1}-1}{p_j-1}$ by the formula for summing geometric progressions, the stated result follows.

Q.766 Both $d(n)$ and $\sigma(n)$ obviously vary greatly from one value of n to the next. (For example, compare $d(n)$ and $\sigma(n)$ when n is 359 or 360). But if one averages the values of $d(n)$ or $\sigma(n)$ for $n = 1, 2, \cdots, N$ a less chaotic behaviour can be observed. Let $[x]$ denote the greatest integer not larger than x;

e.g. $[8] = [8.34..] = 8$

(a) Show that $\sum_{n=1}^{N} d(n) = \sum_{k=1}^{N} [\frac{N}{k}]$

and $\sum_{n=1}^{N} \sigma(n) = \sum_{k=1}^{N} k[\frac{N}{k}]$.

(b) Deduce that $\frac{N}{2} < \frac{1}{N}\sum_{n=1}^{N} \sigma(n) \le N$

and that $2-\frac{1}{N} \le \frac{1}{N}\sum_{n=1}^{N} d(n) \le 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{N} < \log N + 1$ for $N \ge 2$ (using the result of Q.768.)

ANSWER Each natural number k is a factor of $[\frac{N}{k}]$ numbers not exceeding N; viz, of $k, 2k, 3k, \cdots, [\frac{N}{k}]\, k$. Hence k makes a contribution of 1 to each of the $[\frac{N}{k}]$ terms $d(k), d(2k), \cdots, d([\frac{N}{k}]\, k)$ on the L.H.S. of the first identity, and a contribution of k to each of the $\left[\frac{N}{k}\right]$ terms $\sigma(k),\ \sigma(2k), \cdots, \sigma(\left[\frac{N}{k}\right] k)$ of the second identity.

Since this is true for every k both results follow.

(b) For any $x, [x] \leq x$.

$$\therefore \left[\frac{N}{k}\right] \leq \frac{N}{k} \text{ and } k\left[\frac{N}{k}\right] \leq N$$

Also since 1 and n are divisors of n, $d(n) \geq 2$ (except that $d(1) = 1$) and $\sigma(n) > n$ (except that $\sigma(1) = 1$).

Hence $\dfrac{N(N+1)}{2} = \displaystyle\sum_{n=1}^{N} n \leq \sum_{n=1}^{N} \sigma(n) \leq \sum_{n=1}^{N} N = N^2$ for all n

$$\therefore \frac{N}{2} < \frac{N}{2} + \frac{1}{2} \leq \frac{1}{N}\sum_{n=1}^{N}\sigma(n) \leq N$$

Again $2N - 1 = 1 + \displaystyle\sum_{n=2}^{N} 2 \leq \sum_{n=1}^{N} d(n) \leq \sum_{n=1}^{N} \frac{N}{k} = N(1 + \frac{1}{2} + \cdots + \frac{1}{N})$

$$\therefore 2 - \frac{1}{N} \leq \frac{1}{N}\sum_{n=1}^{N} d(n) \leq 1 + \frac{1}{2} + \cdots + \frac{1}{N}$$

($< 1 + A = 1 + \log N$ from Q.768.)

Q.767. It was first proved by L. Euler that the limit sum of the infinite series

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \cdots$$

is equal to $\frac{\pi^2}{6}$.

Assuming this, calculate the sum of

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots + \frac{1}{(2n-1)^2} + \cdots$$

and of $\frac{1}{1^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{13^2} + \frac{1}{17^2} + \frac{1}{19^2} + \frac{1}{23^2} + \frac{1}{25^2} + \cdots$

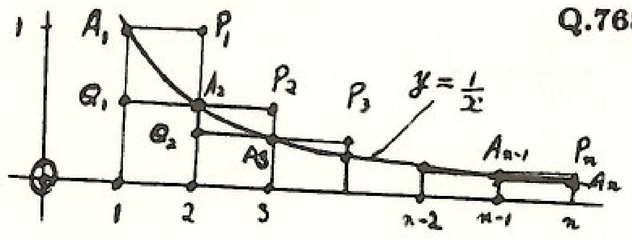
where there remain on the denominators the squares of whole numbers which are relatively prime to 6.

ANSWER

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots+\frac{1}{(2n-1)^2}+\cdots=\left(\frac{1}{1^2}+\frac{1}{2^2}\cdots+\right)$$
$$-\left(\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}\cdots\right)$$
$$=\frac{\pi^2}{6}-\frac{1}{4}\left(\frac{1}{1^2}+\frac{1}{2^2}\cdots\right)$$
$$=\frac{\pi^2}{6}-\frac{\pi^2}{4\times 6}=\frac{\pi^2}{8}$$

Similarly, either $\left(1-\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{6^2}\right)\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}\cdots\right)$

$$=\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\frac{1}{7^2}+\cdots\right)-\left(\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\cdots\right)$$
$$-\left(\frac{1}{3^2}+\frac{1}{6^2}+\frac{1}{9^2}+\cdots\right)+\left(\frac{1}{6^2}+\frac{1}{12^2}+\frac{1}{18^2}+\cdots\right)$$
$$=\frac{1}{1^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots=S$$
$$S=\left(1-\frac{1}{4}-\frac{1}{9}+\frac{1}{36}\right)\times\frac{\pi^2}{6}=\frac{\pi^2}{9}$$
$$\text{or } S=\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\frac{1}{11^2}+\cdots\right)-\left(\frac{1}{3^2}+\frac{1}{9^2}+\frac{1}{15^2}+\cdots\right)$$
$$=\frac{\pi^2}{8}-\frac{1}{3^2}\frac{\pi^2}{8}=\frac{\pi^2}{9}.$$



Q.768. Let $S_n=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$ and let A denote the area in the first quadrant lying under the graph of $y=\frac{1}{x}$ between $x=1$ and $x=n$. Show that $A<S_n<A+1$.

(By calculus, it is shown that $A=\int_1^n \frac{1}{x}dx=\log_e n$.)

ANSWER Refer to the figure. Clearly A is less than the sum of the areas of the taller rectangles (of which the first is 1 2 P_1 A_1)

The area of the jth such rectangle is $1 \times \frac{1}{j}$.

$\therefore A < \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n-1} < S_n$. Likewise A exceeds the sum of the areas of the shorter rectangles (of which the first is $1\ 2\ A_2\ Q_1$). The area of the jth such rectangle is $1 \times \frac{1}{j+1}$. $\therefore$ $A > \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = S_n - 1$. From this $S_n < A + 1$.

Q.769. There are 25 pupils in a class. Of these 18 take French, 14 take History and 9 take Biology, but only one takes all 3 of those subjects. English is a compulsory subject and everyone taking French, History or Biology gained a grade of either B or C in the English exam. There were 5 pupils in the class whose English mark was a D.

How many in the class obtained an A for English?

How many take both History and Biology?

ANSWER There are at most 25-5 pupils in the class who obtained C or better in English, and these 20 include all the pupils who take at least one of F, H, and B. One pupil takes all three of these subjects. If x of the remaining 19 take two of them, and y take only one we have

$$3 \times 1 + 2 \times x + 1 \times y = \#F + \#H + \#B = 18 + 14 + 9 = 41$$

$$\therefore \quad 2x + y = 38 \qquad (1)$$

$$\text{and } x + y \leq 19 \qquad (2)$$

From (2) $2x + 2y \leq 38$ and subtracting (1) yields $y \leq 0$.

But since y is a natural number, $y = 0$ and (1) then solves to give $x = 19$.

Thus no-one takes only one of the three subjects.

Let $u = \#(F \cap H \cap \bar{B})$ (the number who take French and History, but not Biology), $v = \#(F \cap B \cap \bar{H}), w = \#(H \cap B \cap \bar{F})$.

$$\text{Then } u + v = 18 - 1 = 17$$

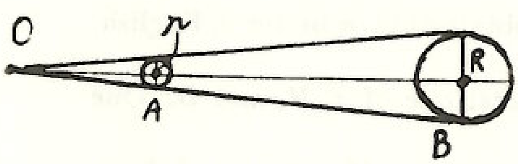
$$u + w = 14 - 1 = 13$$

$$v + w = 9 - 1 = 8$$

$\therefore$ The number who take both History and Biology $= w + 1 = 3$. (The 1 is of course the pupil who takes all three subjects.) Since 5 obtained D in English and the 20 pupils who take French and/or History and/or Biology all obtained B or C, no-one in the class obtained an A in English.

Q.770 The distance to the sun is about 387 times the distance to the moon. Find approximately the ratio of the volume of the sun to that of the moon.

ANSWER As is evident at times of total solar eclipse, it happens that the sun and the moon subtend almost precisely the same angle at a point on the earth. Hence if the radii of sun and moon are R and r respectively $\frac{R}{r} = \frac{OB}{OA} \simeq 387$ to a fair degree of accuracy.



$$\therefore \frac{V\text{sun}}{V\text{moon}} = \frac{\frac{4}{3}\pi R^3}{\frac{4}{3}\pi r^3} \simeq 387^3 \simeq 5.80 \times 10^7$$

Q.771 Consider the sequence of ordered pairs of whole numbers (1,2); (4,9); (17,38); (72,161); (305,682); $\cdots$ Denoting the nth pair by (x_n, y_n), the construction rule is $(x_1, y_1) = (1, 2)$.

$$\left.\begin{aligned} x_{n+1} &= 2x_n + y_n \\ y_{n+1} &= 5x_n + 2y_n. \end{aligned}\right\} \text{ for all } n \geq 1.$$

(i) Prove that $x_n \geq 4^{n-1}$ for all n.

(ii) Find a formula for $5x_n^2 - y_n^2$ for all n.

(iii) When n is large $\frac{y_n}{x_n}$ is very close to a certain number α. Find α. How large should n be taken to ensure that $\frac{y_n}{x_n}$ differs from α by less than $\frac{1}{10^6}$?

ANSWER (i) It is clear that x_n and y_n are positive integers for every n, and that

$$y_n = 5x_{n-1} + 2y_{n-1} + 2y_{n-1} = 2(2x_{n-1} + y_{n-1}) + x_{n-1}$$

$$= 2x_n + x_{n-1} \geq 2x_n \text{ for } n > 1.$$

$$\therefore x_{n+1} = 2x_n + y_n \geq 2x_n + 2x_n = 4x_n \text{ for } n > 1.$$

Since $x_1 = 1$, we deduce that $x_2 \geq 4 \times 1 = 4^1$,, then $x_3 \geq 4 \times 4^1 = 4^2$, and so on. Thus $x_n \geq 4^{n-1}$ for all positive integers n.

(ii) $5x_{n+1}^2 - y_{n+1}^2 = 5(2x_n + y_n)^2 - (5x_n + 2y_n)^2 = -5x_n^2 + y_n^2$

$$= -(5x_n^2 - y_n^2) \text{ for all } n \geq 1$$

Since $5x_1^2 - y_1^2 = 5 \times 1 - 2^2 = 1$, we deduce that $5x_2^2 - y_2^2 = -1$, $(5x_3^2 - y_3^2) = (-1) \times (-1) = 1$, and so on. Thus $5x_n^2 - y_n^2 = (-1)^{n-1}$ for all positive integers n.

(iii) From this, $\left|5 - \left(\frac{y_n}{x_n}\right)^2\right| = \left|\frac{(-1)^{n-1}}{x_n^2}\right| \leq \frac{1}{(4^{n-1})^2}$

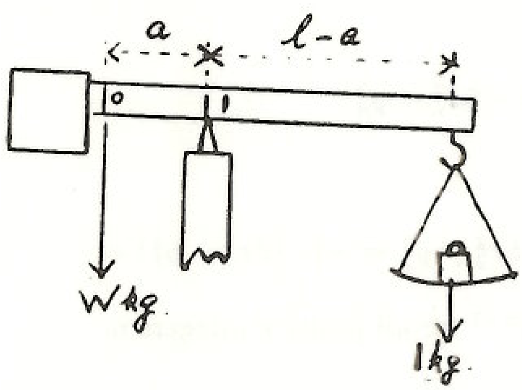
$\left|(\sqrt{5} - \frac{y_n}{x_n})\right|(\sqrt{5} + \frac{y_n}{x_n}) \leq \frac{1}{2^{4n-4}}$

One deduces that $\left|\frac{y_n}{x_n} - \sqrt{5}\right|$ must become very small as n increases so that $\frac{y_n}{x_n}$ becomes very close to $\sqrt{5}$. Then since $\left(\frac{y_n}{x_n} + \sqrt{5}\right) \simeq 2\sqrt{5} > 4$, we will have $\left|\frac{y_n}{x_n} - \sqrt{5}\right| < \frac{1}{2^{4n-2}}$. Since $2^{20} > 10^6$, $\frac{y_n}{x_n}$ will differ from $\sqrt{5}$ by less than $\frac{1}{10^6}$ if $4n - 2 \geq 20$. Provided n is at least 6 this degree of accuracy will be achieved.

(Checking with a calculator, $|\sqrt{5} - y_5/x_5| = |\sqrt{5} - \frac{682}{305}| \simeq .0000024$,

and $|\sqrt{5} - y_6/x_6| = |\sqrt{5} - \frac{2889}{1292}| \simeq .0000001$.)

Q.772 A primitive balance consists of a straight length of material attached at one end to a heavy block. At the other end is a scale pan suspended from a cuphook. A mark labelled 0 is placed on the rod at the balance point when there is no weight on the scalepan. How can one calibrate the instrument if one has only a single one kgm weight, a tape measure and a pen.



ANSWER Make a second mark, labelled "1", on the rod at the point of balance when the 1 kgm weight is on the scale pan. Measure the distances between the marks "0" and "1" (a cms), and from "0" to the cuphook (ℓ cms). If W (kgms weight) denotes the weight of the block, rod and scalepan (which can be regarded as acting vertically through the mark "0") then the moments equation when the 1km weight is balanced is

$$W \times a = 1 \times (\ell - a). \tag{1}$$

Let x_y denote the distance from "0" to the balance point when y kgms is in the scalepan. The moments equation is now

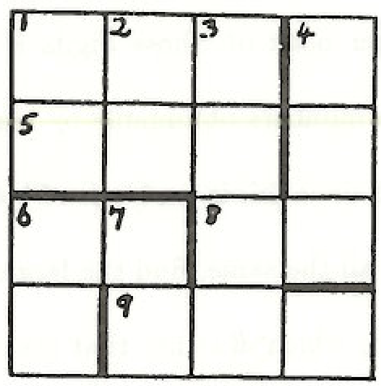
$$W \times x_y = y \times (\ell - x_y). \tag{2}$$

Eliminating W between (1) and (2) and solving for x_y yields

$$x_y = \frac{a\ell y}{\ell + a(y-1)}.$$

A mark labelled "y" can be made at the point corresponding to this calculation for each of any desired set of values of y to calibrate the instrument.

CROSS NUMBER PUZZLE



Four of the numbers are perfect cubes and three are perfect squares. 1 across is a square. 1 down is a prime factor of 6 across and of 7 down. 3 down whose digits add up to fifteen, is the product of 1 down and 4 down. 2 down, which is a factor of 9 across, is one less than 7 down. No two of the eleven numbers are the same.

ANSWER: p40