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Solutions 1711–1720

Q1711 The numbers $1, 2, \ldots, 64$ are written onto the squares of an 8×8 chessboard, one to a square. In problem 640 (*Parabola* volume 21, issue 2 (1985)) it was proved that there must be two adjacent squares (sharing a side or a corner) which contain numbers differing by 16 or less.

(a) Find such an arrangement in which the minimum difference between numbers in squares which are adjacent horizontally, vertically or diagonally is 15.

(b) Prove that there is no such arrangement in which the minimum difference is greater than 15.

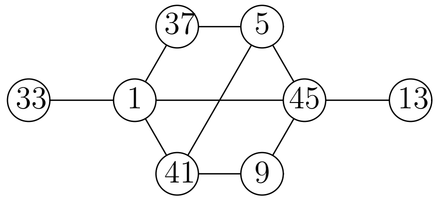
SOLUTION Finding an arrangement of the numbers $1, 2, \ldots, 64$ on a chessboard in which the minimum difference between adjacent numbers is 15 is to a large extent a matter of trial and error. However, intelligent trial and error is always better than mindless trial and error! – so we discuss some ideas before giving an answer. We begin by placing the following numbers in a row of eight squares:

1	17	2	18	3	19	4	20

The rationale for doing this is that we have a row where differences between adjacent numbers are 15 or 16, while using only a selection from the smallest numbers available: in this way we may hope to use larger numbers in the next row, and thereby avoid having any differences less than 15 between the two rows. Cataloguing similar rows of the same form to use all numbers up to 32, and then beginning again at 33, we have

1	17	2	18	3	19	4	20
5	21	6	22	7	23	8	24
9	25	10	26	11	27	12	28
13	29	14	30	15	31	16	32
33	49	34	50	35	51	36	52
37	53	38	54	39	55	40	56
41	57	42	58	43	59	44	60
45	61	46	62	47	63	48	64

Now it is obvious that this arrangement includes many differences less than 15 and therefore does not solve our problem; but perhaps we can succeed by using the same rows in a different order? It is clear that the row beginning with 1 cannot be next to those beginning with 5, 9 or 13; a careful examination shows that it *can* be next to those beginning with $33, 37, 41$ or 43. Considering other rows in the same way, we find that the possible adjacencies between rows are given by the following diagram.



We need to find in this diagram a path connecting all eight entries without repetition, and there is little difficulty in finding that the only possibility is 33–1–37–5–41–9–45–13, or its reverse. Therefore the following placement of numbers meets our requirements.

33	49	34	50	35	51	36	52
1	17	2	18	3	19	4	20
37	53	38	54	39	55	40	56
5	21	6	22	7	23	8	24
41	57	42	58	43	59	44	60
9	25	10	26	11	27	12	28
45	61	46	62	47	63	48	64
13	29	14	30	15	31	16	32

To tackle part (b), suppose that we have placed the sixty–four numbers in such a way that the minimum difference between adjacent numbers is 16 or more: that is, adjacent numbers never differ by 15 or less. Divide the chessboard into sixteen 2×2 "cells". No two of the sixteen numbers $1, 2, \ldots, 16$ can occur in the same cell, as this would give two adjacent squares with a difference of 15 or less. So each must be in a different cell. Label the cells $1, 2, \ldots, 16$ according to which of these numbers it contains: so, cell k will contain the number k, and three further numbers from $17, 18, \ldots, 64$. Which cell contains the number 17? It cannot be cell k for $k = 2, 3, \ldots, 16$, as there would then be adjacent squares with difference $17 - k < 16$; so it must be cell 1. The number 18 cannot be in cells $3, 4, \ldots, 16$ for a similar reason; nor can it be in cell 1, as we now know that that cell also contains 17; so it is in cell 2. Continuing in this way for the numbers $19, 20, \ldots, 32$, and then similarly for $33, 34, \ldots, 48$ and then $49, 50, \ldots, 64$, we see that the numbers in cell k are

$$C_k = \{ k, \ k + 16, \ k + 32, \ k + 48 \} .$$

Now consider any of the four cells *not* meeting the boundary of the chessboard, say, cell m. The number m in this cell is on the same side of the cell as either $m + 32$ or $m + 48$ (perhaps both). In either case, m and the other number are not on the boundary of the chessboard, and hence are both adjacent to two numbers x, y in an adjoining cell.

(a) Suppose the numbers referred to are m and $m+32$. Then x and y cannot be from 1 to $m+15$, as all of these numbers are within 15 of m; they cannot be $m+16$, as that is in C_m; they cannot be from $m+17$ to $m+47$, as these are within 15 of $m+32$; and they cannot be $m+48$ as that is in C_m. Therefore both x and y belong to

$$\{\, m+49,\ m+50, \ldots, 64\,\} ,$$

a set of at most 15 consecutive integers; so x and y differ by less than 16, and this possibility must be ruled out.

(b) If the numbers referred to are m and $m+48$, then for similar reasons, x and y cannot be from 1 to $m+15$, nor $m+16$, nor $m+32$, nor from $m+33$ to 64; so they both belong to

$$\{\, m+17,\ m+18, \ldots, m+31\,\} ,$$

and this again must be ruled out.

There are no options left, and we conclude that placing the numbers with minimum difference 16 or more between adjacent squares is impossible.

Q1712 Let $f(x)$ be a cubic polynomial with leading coefficient 4, say,

$$f(x) = 4x^3 + a_2x^2 + a_1x + a_0 ,$$

having the property that $|f(x)| \le 1$ whenever $|x| \le 1$. Prove that the only possible such polynomial is $f(x) = 4x^3 - 3x$. (We showed in Problem 523, *Parabola* Volume 18 Issue 1 (1982), that no cubic with leading coefficient greater than 4 has the stated property.)

SOLUTION Let

$$f(x) = 4x^3 - 3x + g(x) ,$$

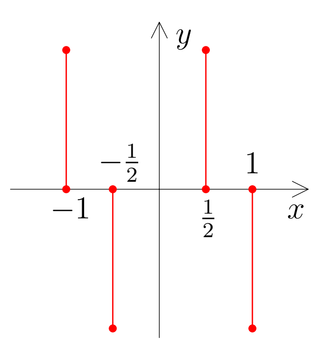
where $g(x)$ is a polynomial of degree at most 2. Using the stated property of $f(x)$ and taking $x = 1$, we have

$$-1 \le 1 + g(1) \le 1 \quad \text{and so} \quad -2 \le g(1) \le 0 .$$

In the same way, we find

$$0 \le g\big(\tfrac12\big) \le 2 , \quad -2 \le g\big(-\tfrac12\big) \le 0 , \quad 0 \le g(-1) \le 2 .$$

Therefore the graph of $y = g(x)$ must pass through each of the line segments marked in red,



and this is impossible for a quadratic or linear polynomial. Therefore $g(x)$ is a constant; since $g(1) \leq 0$ and $g(-1) \geq 0$, this constant must be zero. Therefore

$$f(x) = 4x^3 - 3x \ ,$$

as claimed.

To give the "impossibility" argument more explicitly, suppose that $g(x)$ is not constant. Then it is decreasing between $x = -1$ and $x = -\frac{1}{2}$ (perhaps not always: it could be increasing briefly and then decreasing: but it is certainly decreasing *somewhere* in this interval). It is increasing between $x = -\frac{1}{2}$ and $x = \frac{1}{2}$, and decreasing again between $x = \frac{1}{2}$ and $x = 1$. But a quadratic cannot be decreasing–then–increasing–then–decreasing: it can only be increasing–then–decreasing, or *vice versa*. And a linear polynomial is always increasing, or always decreasing. So the only possibility is that $g(x)$ is a constant polynomial.

Q1713 In *Parabola* Problem 1240 (Volume 43 Issue 1 (2007)) we proved the following fact about the set of integers $S = \{0, 1, 2\}$: if $f(x) = ax^2 + bx + c$ is any quadratic with real coefficients such that $f(x)$ is an integer for all values of x in the set S, then $f(x)$ is an integer for all integer values of x.

(a) Find all possible sets of three integers which have the same property.

(b) Can you find a set of four or more integers which does not include S or any of the other sets you found in (a), and which still has the same property?

SOLUTION Recall that we are investigating sets S of integers having the following property for all real quadratics $f(x) = ax^2 + bx + c$:

> "if $f(x)$ is an integer for all x in S,
> then $f(x)$ is an integer for all integers x".

Since we are looking at Integer values of Quadratics, we shall call such a set an *IQ–set*. We already know that $S = \{0, 1, 2\}$ is an IQ–set, and our first task is to find all possible IQ–sets of three integers. We approach the problem by stages.

- Step 1: if $S = \{0, 1, s\}$ with $s > 2$, then S is not an IQ–set. **Proof**. The quadratic

  $$f(x) = \frac{x^2 - x}{s(s-1)}$$

  is an integer for $x = 0, 1, s$, as is easily checked; however,

  $$f(2) = \frac{2}{s(s-1)}$$

  is not an integer, because the denominator has a factor $s > 2$.

- Step 2: if $S = \{0, s, t\}$ with $1 < s < t$, then S is not an IQ–set. **Proof**. Similar to the above: consider
$$f(x) = \frac{x^2 - sx}{t(t-s)}$$
and check that while $f(0)$, $f(s)$ and $f(t)$ are integers, $f(1)$ is not.

- Step 3: it follows from steps 1 and 2 that the only three–element IQ–set in which the smallest element is 0 is our original example $\{0, 1, 2\}$. To finish the problem, we show that any other IQ–set of three elements is closely related to this one.

- Step 4. Let $S = \{s, t, u\}$ and $T = \{s-k,\ t-k,\ u-k\}$, where k is any integer. Then T is an IQ–set if and only if S is. **Proof**. Suppose that S is an IQ–set; we aim to show that T is an IQ–set. Let $f(x)$ be any real quadratic such that $f(s-k)$, $f(t-k)$ and $f(u-k)$ are integers; we have to prove that $f(x)$ is an integer for all integers x. The assumption that S is an IQ–set means that the stated property applies to *any* quadratic: so we may apply it to the quadratic $g(x) = f(x-k)$. This is an integer for $x = s, t, u$; so $g(x)$ is an integer for all integers x; so $g(x+k) = f(x)$ is an integer for all integers x. Thus T is an IQ–set. We also need to show that if T is an IQ–set then so is S; the argument for this is virtually identical and will be omitted.

We've done all the hard work, and it is now relatively easy to find all three–element IQ–sets. Let $S = \{s, t, u\}$ with $s < t < u$. Then

$$\begin{aligned} & S \text{ is an IQ–set} \\ \Leftrightarrow\quad & T = \{0, t-s, u-s\} \text{ is an IQ–set} \quad [\text{step } 4, \text{ taking } k = s] \\ \Leftrightarrow\quad & t-s = 1, u-s = 2 \qquad [\text{step } 3] \\ \Leftrightarrow\quad & S = \{s, s+1, s+2\}. \end{aligned}$$

That is, a three–element set has the property we seek if and only if it consists of three consecutive integers.

It is not hard to see that if a set contains three consecutive integers and more besides, this does not affect the IQ–property. However, it is also the case that $S = \{0, 1, 3, 5\}$ is an IQ–set, despite the fact that it does not contain three consecutive integers. To prove this, consider the quadratic $f(x) = ax^2 + bx + c$, and suppose that $f(0),\ f(1),\ f(3),\ f(5)$ are all integers. That is,

$$c\,, \quad a + b + c\,, \quad 9a + 3b + c\,, \quad 25a + 5b + c$$

are integers. It follows that $a + b = (a + b + c) - c$ is the difference of two integers and hence is itself an integer; and likewise that

$$2a = (25a + 5b + c) - 3(9a + 3b + c) + 4(a + b + c) - 2c$$

is an integer. If $2a$ is even then a is an integer, so is b, and hence $f(x) = ax^2 + bx + c$ is an integer whenever x is an integer. If $2a$ is odd then so is $2b$, because $2(a + b)$ is even.

Then we can write

$$f(x) = \Big(a - \frac{1}{2}\Big)x^2 + \Big(b - \frac{1}{2}\Big)x + c + \frac{x(x+1)}{2}\ .$$

Let x be an integer. The coefficients in brackets are integers; and $x(x+1)$ is even, so the last term is an integer; so $f(x)$ is an integer. Thus $\{\,0, 1, 3, 5\,\}$ is an IQ–set.

Q1714 In Problem 1089 (*Parabola* Volume 36 Issue 3 (2000)), we described an apartment block consisting of 120 apartments. Every day, the inhabitants of an "impossible" apartment – one having 15 or more residents – all go off to other apartments in the same block, no two to the same apartment. We showed that if there is a total of 119 residents, then sooner or later there will be no more impossible apartments. Now suppose that one more person moves in, so that there are 120 residents in all. Devise a scenario in which there will always be an impossible apartment.

SOLUTION The solution to problem 1089 (look it up for full details) relied on counting the number of pairs of people inhabiting the same apartment: if the n residents of an impossible apartment move off to n different apartments where the current numbers of residents are $a_1, a_2, \ldots, a_n$, then the number of pairs decreases by $C(n,2) = n(n-1)/2 \geq 105$ and increases by $a_1 + a_2 + \cdots + a_n \leq 119 - n \leq 104$, so the overall number of pairs decreases; this cannot go on for ever.

If we apply the same ideas to the case of 120 residents, we shall have

$$C(n,2) \geq 105 \quad \text{and} \quad a_1 + a_2 + \cdots + a_n \leq 120 - n \leq 105\ .$$

This will still result in a decrease in the number of pairs, *unless* all of these inequalities become equalities. So we will need one apartment having exactly 15 residents, and 15 other apartments having a total of 105 residents. One way of ensuring that "impossible" apartments persist indefinitely is to devise a scenario in which the numbers of occupants in various apartments are the same after a redistribution as they were before. So, suppose that all but 16 apartments are absolutely uninhabitable and always remain empty; and that the sixteen liveable apartments house

$$15,\ 14,\ 13, \ldots,\ 2,\ 1,\ 0$$

residents respectively. Then the residents of the impossible apartment redistribute themselves to the other 15 of these 16 apartments, one to each, so that the numbers of residents become

$$0,\ 15,\ 14, \ldots,\ 3,\ 2,\ 1.$$

Since these numbers are exactly the same as they were before, they will also be the same after the next redistribution; and so on indefinitely. Hence, there will always be an apartment with 15 or more residents.

Q1715 In Problem 666 from *Parabola* Volume 22 Issue 1 (1984), we showed that if A is any finite, non–empty subset of

$$S = \{\,2,\,2^3,\,2^5,\dots\}\ ,$$

then the sum of the elements of A cannot be a perfect square; and we also did something similar for cubes.

Now find an infinite set S of positive integers such that for any finite non–empty subset A of S, the sum of all elements of A is never a perfect power. By a *perfect power* we mean a positive integer a^b, where a, b are positive integers and $b > 1$.

SOLUTION The key features of the set S above are that each number is (a multiple of) the prime $p = 2$ to an odd power; that each number is a factor of all larger numbers in the set; and that in each case the quotient has the same prime as a factor. As we have seen, we can also make this idea work for cubes; but doing so makes it fail for squares; and the same holds for higher powers. (Nevertheless, a very similar construction will work, for instance, for squares and cubes simultaneously: we leave this for readers to think about.)

To construct a set which will have the desired property for all powers simultaneously, we write $p_1, p_2, \dots$ for the prime numbers, define

$$a_k = p_1^2 p_2^2 \cdots p_{k-1}^2 p_k\ ,$$

and let S be the set of all a_k with $k \ge 1$. Explicitly,

$$\begin{aligned} S &= \{\,2,\,2^2{\times}3,\,2^2{\times}3^2{\times}5,\,2^2{\times}3^2{\times}5^2{\times}7,\dots\} \\ &= \{\,2,\,12,\,180,\,6300,\,485100,\,69369300,\dots\}\ . \end{aligned}$$

For any k we have the following important properties: p_k is a factor of a_k, but p_k^2 is not a factor of a_k; also, a_k is a factor of all larger elements of S, and the quotient is a multiple of p_k. So, take any subset

$$A = \{\,a_{m_1},\,a_{m_2},\,\dots,\,a_{m_n}\,\}$$

of S, where we assume, as we may, that m_1 is the smallest of all the subscripts. The sum of these elements is

$$N = a_{m_1} + a_{m_2} + \cdots + a_{m_n} = a_{m_1}\left(1 + \frac{a_{m_2}}{a_{m_1}} + \cdots + \frac{a_{m_n}}{a_{m_1}}\right)\ .$$

By our previous remarks, all the quotients a_{m_k}/a_{m_1} for $k > 1$ are integers, and are multiples of p_{m_1}. So the expression in brackets is 1 plus a sum of multiples of p_{m_1}, and therefore is not a multiple of p_{m_1}. (Note that this is still true even if there are no terms in the brackets after the initial 1.) Hence, the only factor p_{m_1} in the prime factorisation of N is the single p_{m_1} in a_{m_1}. In the prime factorisation of a bth power, every prime must have an exponent which is a multiple of b; but that of N has p_{m_1} with exponent 1, and so N cannot be a bth power with $b > 1$.

Q1716 A bank safe has a security keypad consisting of buttons which bear the digits 0 to 9; a password to gain access to the safe consists of six digits (with repeated digits being permitted). Because of the keypad's innovative design, a valid password must be such that the maximum difference between any two of its digits is exactly equal to n, a number from 1 to 9; it is up to the bank's security office to decide what value of n should be specified. Eventually they decide that in order to provide the maximum different number of passwords, they will just go with $n = 9$. Is this a wise choice?

SOLUTION Consider strings of six digits chosen from 0 to n. There are $(n+1)^6$ such strings, but not all will be valid passwords: if either of the digits 0 and n is omitted, then the maximum difference will be less than n, rather than equal to n as required. We shall count the valid strings by using the principle of inclusion/exclusion: if the complete set of strings (chosen from 0 to n) is T, the set omitting 0 is A, the set omitting n is B and the set omitting both is C, then the number of valid passwords is

$$|T| - |A| - |B| + |C| = (n+1)^6 - 2n^6 + (n-1)^6 \ .$$

There would be the same number of valid passwords with digits from 1 to $n+1$; and from 2 to $n+2$; and, eventually, from $9-n$ to 9. So there are $10-n$ possible cases, and the total number of valid passwords is

$$(10-n)\big((n+1)^6 - 2n^6 + (n-1)^6\big) \ .$$

Taking $n = 9$, the bank will allow 199262 passwords. However, with $n = 8$ there would have been 249604 passwords, so the security office did not make a good choice.

Note. If you are ever in charge of organising security in real life, make sure that you have **heaps** more potential passwords than this!

Q1717 Find the smallest perfect square whose decimal representation consists of the same block of digits twice over. (An example of such a number would be 123123 – but of course, that's not a square.) As usual, the first digit of a number may not be zero.

SOLUTION Suppose that a square s^2 has digits $d_1d_2\cdots d_nd_1d_2\cdots d_n$, and let x be the number with digits $d_1d_2\cdots d_n$. Then we have

$$s^2 = x + 10^n x = (10^n + 1)x \ .$$

Let a^2 be the largest square factor of $10^n + 1$; then we can write

$$10^n + 1 = a^2 b \quad \text{and so} \quad s^2 = a^2 bx \ ,$$

where b is an integer with no square factor other than 1. For the right–hand side to be a square, x must have the same prime factors as b, once each; and any additional prime factors of x must occur an even number of times. That is, we have

$$x = bc^2$$

for some integer c. Now since x has n digits,

$$b \le x < 10^n < 10^n + 1 = a^2 b \ ,$$

and so a must be greater than 1. With computing assistance we can find the factorisations of 11, 101, 1001, 10001 and so on; it turns out that the first number in this sequence to have a square factor greater than 1 is

$$10^{11} + 1 = 100000000001 = 11^2 \times 826446281 ,$$

and so we have

$$n = 11 , \quad a = 11 , \quad b = 826446281 .$$

Using again the fact that x has n digits (not beginning with a zero), we have $10^{n-1} \leq x < 10^n$ and hence

$$\frac{10^{10}}{826446281} \leq c^2 < \frac{10^{11}}{826446281} ,$$

that is, $13 \leq c^2 \leq 120$; to obtain the smallest possible value for x, we take $c^2 = 16$, which yields $x = 13223140496$. So the smallest square consisting of two identical digit–strings is

$$1322314049613223140496 ,$$

which you can confirm is equal to 36363636364^2.

Solution received from Hyunbin Yoo, South Korea.

Q1718 Let a, b, c be positive numbers. Prove that if $\sqrt{a} + \sqrt{b} + \sqrt{c} \leq 4$, then

$$\frac{a}{7 + a^4} + \frac{b}{7 + b^4} + \frac{c}{7 + c^4} \leq \frac{1}{2} .$$

SOLUTION We use the Arithmetic–Geometric Mean inequality for eight numbers:

$$\frac{x_1 + x_2 + \cdots + x_8}{8} \geq \sqrt[8]{x_1 x_2 \cdots x_8} .$$

Taking $x_1 = x_2 = \cdots = x_7 = 1$ and $x_8 = a^4$, and multiplying by 8 on both sides of the result, this becomes

$$7 + a^4 \geq 8\sqrt{a} ,$$

which then gives

$$\frac{a}{7 + a^4} \leq \frac{a}{8\sqrt{a}} = \frac{\sqrt{a}}{8} .$$

Doing the same for b and c, and adding the three results, we have

$$\frac{a}{7 + a^4} + \frac{b}{7 + b^4} + \frac{c}{7 + c^4} \leq \frac{\sqrt{a}}{8} + \frac{\sqrt{b}}{8} + \frac{\sqrt{c}}{8} \leq \frac{1}{2} .$$

Solutions received from Toyesh Prakash Sharma (who also contributed the problem) and from Henry Ricardo, New York, USA.

Alternative solution from Titu Zvonaru, Comăneşti, Romania. Using the Arithmetic–Geometric Mean inequality for four numbers, we have

$$7 + a^4 > x^4 + 2 + 2 + 2 \ge 4\sqrt[4]{8a^4} = \bigl(4\sqrt[4]{8}\bigr)a\ ,$$

and likewise for b and c. It is not hard to see that $8 > \frac{81}{16} = \bigl(\frac{3}{2}\bigr)^4$, and so

$$\frac{a}{7+a^4} + \frac{b}{7+b^4} + \frac{c}{7+c^4} < \frac{3}{4\sqrt[4]{8}} < \frac{1}{2}\ .$$

Q1719 Find all integers $n > 0$ for which $n^2 + 2023$ and $(n+1)^2 + 2023$ have a common factor greater than 1.

SOLUTION If d is a common factor of $n^2 + 2023$ and $(n+1)^2 + 2023$, then we see successively that it is also a factor of

$$\begin{gathered}\bigl((n+1)^2 + 2023\bigr) - \bigl(n^2 + 2023\bigr) = 2n + 1\ ;\\ n(2n+1) - 2(n^2 + 2023) = n - 4046\ ;\\ (2n+1) - 2(n - 4046) = 8093\ .\end{gathered}$$

But 8093 is prime; therefore, since we want $d > 1$, the only possibility is $d = 8093$. Thus $n - 4046$ is a multiple of 8093, that is,

$$n = 4046 + 8093k \qquad (*)$$

for some integer k. Now at this stage it is possible that this only works for certain values of k, maybe even for none at all: so we have to check that our above working is also valid in reverse. If $(*)$ holds, then 8093 is successively a factor of

$$\begin{gathered}n - 4046\ ;\\ 2(n - 4046) + 8093 = 2n + 1\ ;\\ n(n - 4046) + 2023(2n+1) = n^2 + 2023\ ;\end{gathered}$$

and finally,

$$(n^2 + 2023) + (2n + 1) = (n+1)^2 + 2023\ .$$

Therefore the positive integers n for which $n^2 + 2023$ and $(n+1)^2 + 2023$ have a common factor greater than 1 are all those given by $(*)$ with k a non–negative integer (and the common factor is 8093).

Q1720 Prove that if $a_1, a_2, a_3, \ldots, a_n$ are positive numbers, and M is a positive constant such that

$$a_1 < M^2\ , \quad a_2 < M^4\ , \quad a_3 < M^8\ , \ldots, \quad a_n < M^{2^n}\ ,$$

then

$$\sqrt{a_1 + \sqrt{a_2 + \sqrt{a_3 + \cdots + \sqrt{a_n}}}} < M\sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots + \sqrt{1}}}}\ ,$$

where there are n square root symbols on the right–hand side.

SOLUTION We prove the result by induction. If we have only one number a_1, then our assumption is that it satisfies $a_1 < M^2$, and the required inequality

$$\sqrt{a_1} < M\sqrt{1}$$

is obviously true.

Now suppose that the result is true for some specific integer n: we have to prove it is true for $n+1$. This can be done by simple algebra; however, it's easy to get confused unless we are careful with the notation. So, let $a_1, a_2, a_3, \ldots, a_{n+1}$ be positive numbers and M a constant such that

$$a_k < M^{2^k}$$

for all k. Let $b_1 = a_2$, $b_2 = a_3$ and so on; then we have

$$b_k = a_{k+1} < M^{2^{k+1}}$$

for every k; using power laws carefully, this can be written as

$$b_k < N^{2^k} \ ,$$

where $N = M^2$. Since we have n terms b_k, we can use the inductive assumption to conclude that

$$\sqrt{b_1 + \sqrt{b_2 + \cdots + \sqrt{b_n}}} < N\sqrt{1 + \sqrt{1 + \cdots + \sqrt{1}}}\,,$$

that is,

$$\sqrt{a_2 + \sqrt{a_3 + \cdots + \sqrt{a_{n+1}}}} < M^2\sqrt{1 + \sqrt{1 + \cdots + \sqrt{1}}}$$

with n square roots. The rest is easy: add a_1 to both sides, take square roots and do the algebra, remembering that $a_1 < M^2$:

$$\begin{aligned}\sqrt{a_1 + \sqrt{a_2 + \sqrt{a_3 + \cdots + \sqrt{a_{n+1}}}}} &< \sqrt{M^2 + \underbrace{M^2\sqrt{1 + \sqrt{1 + \cdots + \sqrt{1}}}}_{n \text{ square roots}}} \\ &= M\underbrace{\sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots + \sqrt{1}}}}}_{n+1 \text{ square roots}}\,,\end{aligned}$$

and we are done. By induction, the result claimed is true for any n.

Solution received from Hyunbin Yoo, South Korea.