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Problems 1761–1770

Parabola would like to thank user Evan at `math.stackexchange.com` for contributing Problem 1762.

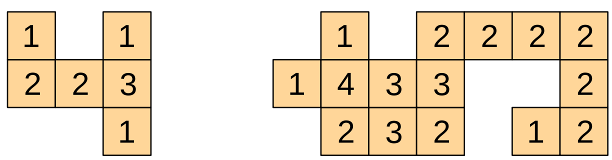
Q1761 Let a be an integer. Find the number of integers b such that the quadratic

$$(x+a)(x+b)+2025$$

can be factorised as the product of two linear factors with integer coefficients.

Comment. Isn't this basically the same as a question we asked last year? Don't be hasty…

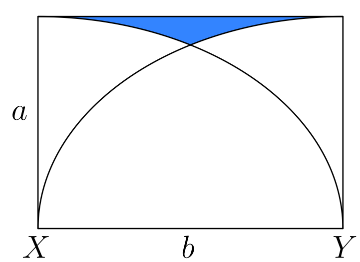
Q1762 A *polyomino* is a connected arrangement of squares, all the same size, and never meeting horizontally or vertically except along the full length of a side. Here are two examples; in each example, every square is labelled with the number of squares which adjoin it.



Is there an arrangement of this type in which the numbers $1, 2, 3, 4$ all occur equally often? If so, then find a smallest possible example.

Thanks to user Evan at `math.stackexchange.com` for inventing this problem and allowing *Parabola* to publish it.

Q1763 The diagram shows an a by b rectangle containing two quarters of ellipses with centres at X and Y. Calculate the area of the blue shaded region.

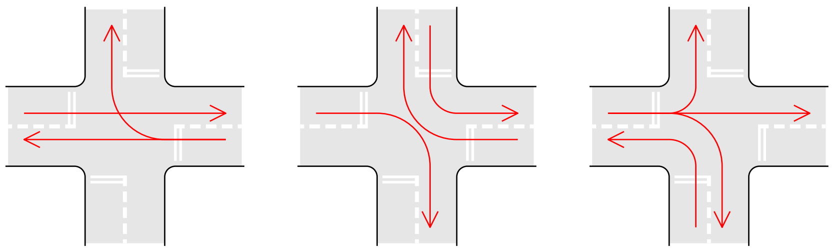


Q1764 We wish to find sequences of positive numbers $x_1, x_2, \ldots, x_{18}$ such that

- for every k from 1 to 17, we have either $x_{k+1} = 4x_k$ or $x_k = 5x_{k+1}$;
- $x_1 + x_2 + \cdots + x_{18} = 2025$.

How many such sequences are there? Show that **none** of the numbers $x_1, x_2, \ldots, x_{18}$ in such a sequence can be an integer.

Q1765 A group of traffic engineers is designing a traffic–light sequence for an intersection of two two–way roads. Any vehicle approaching the intersection will see lights showing whether travel straight on, or left, or right is permitted. (No U–turns!) An allowable traffic flow must be such that there is no possibility of collisions between cars travelling through the intersection at the same time, as long as they are obeying the lights. Three traffic flows are shown below.



The first of these is not allowable because a car turning from the eastern road to the northern could collide with one travelling straight from the western road to the eastern. The second and third traffic flows are OK.

How many allowable traffic flows are there? (**Note**. This puzzle is published in Australia, so cars must drive on the left side of the road!) The identity of roads is important: so for example, rotating the third example above to a different orientation *does count* as a different arrangement.

Q1766 Holly and Ted take turns throwing a coin, with Holly going first. Holly wins if she throws a head; Ted wins if he throws a tail. (Once one of these events occurs, the game is over and there are no further throws.) However, the coin is not known to be a fair coin: that is, the probability p of throwing a head might not be $\frac{1}{2}$. Find the value of p for which Ted has an even chance of winning this game.

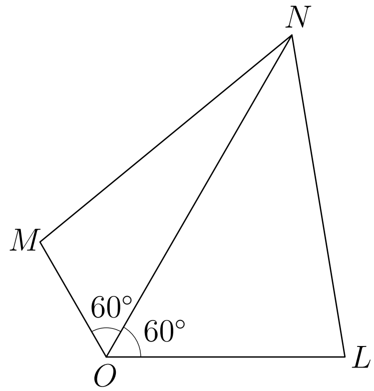
Q1767 Let S be a set of numbers from 0 to 1 which has the following properties.

(1) The numbers 0 and 1 are in S.

(2) Whenever x and y are numbers in S, their average $(x + y)/2$ is also in S.

Find an example in which S **is not** the set of all numbers from 0 to 1. Then prove that if there is a number a with $0 < a < 1$, such that S contains all numbers from a to 1, then S **is** the set of all numbers from 0 to 1.

Q1768 In the diagram, there are angles of $60°$ as marked; also, $ON = OM + OL$. Prove that $\angle MNL = 60°$.



Q1769 (a) Prove that if x is any real number, then

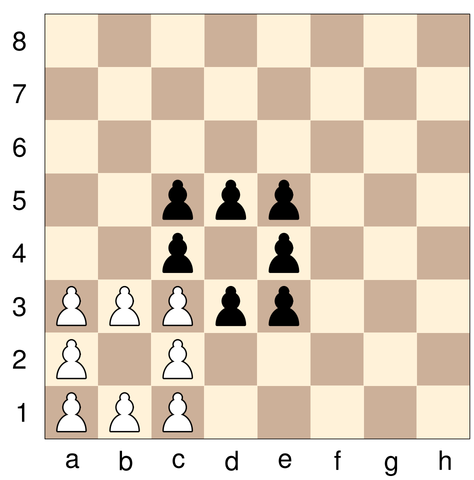
$$\sin 2x \sin 9x + \sin 3x \sin 4x - \sin 5x \sin 6x = 0\,.$$

(b) Prove that there are no positive numbers a, b, c, d, e, f such that

$$\sin ax \sin bx + \sin cx \sin dx + \sin ex \sin fx = 0$$

for all real values of x.

Q1770 Consider the following placement of eight white pawns and seven black pawns on a chessboard. (**Note** that this is not going to be a chess problem, so it does not matter that the position is illegal!)



In the game of chess, a knight can move from its current square to any square reached by going two squares horizontally or vertically, then one square in a perpendicular direction. It captures a pawn by moving onto the pawn's square. In this problem, a grey knight (it is grey so that it can capture both white and black pawns) was placed on the chessboard, on a square not occupied by any pawn; then made 15 consecutive moves capturing all 15 pawns; and then one more move returning to its initial square. The first and last pawns captured by the knight were of the same colour. On which square did the knight start?