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Infinite series identities involving π and $\ln 2$

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In celebration of Pi Day 2025

1 Introduction

In this paper we prove identities for six infinite series whose values involve linear combinations of π and $\ln 2$, that do not appear in standard infinite series references.

2 New formulas for π

In a previous paper in *Parabola* [8] published to celebrate Pi Day 2022, the second author proved the identity

$$\sum_{n=1}^{\infty} \frac{1}{n(2n-1)(4n-3)} = \frac{\pi}{3}\,, \tag{1}$$

as well as a list of twenty-seven further infinite series identities involving linear combinations of $\pi, \pi^2, \ln 2$ and $\ln 3$.

In the closing of [8], one more formula is noted as a problem for the reader, which is to find the value of a similar series absent from the list:

$$\sum_{n=1}^{\infty} \frac{1}{n(4n-1)(4n-3)}\,. \tag{2}$$

In a 2024 undergraduate research project at Michigan Technological University mentored by the second author, the first author of this paper found the value of series (2).

Theorem 1. *We have the identity*

$$\sum_{n=1}^{\infty} \frac{1}{n(4n-1)(4n-3)} = \frac{\pi}{3} - \ln 2\,.$$

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To prove this theorem, we combine entries (15) and (16) of [8]. Let S_{15} and S_{16} denote the infinite series given by these two entries, respectively:[3]

$$\begin{aligned} S_{15} &= \sum_{n=1}^{\infty} \frac{1}{n(4n-3)} = \frac{6\ln 2 + \pi}{6}\,; \\ S_{16} &= \sum_{n=1}^{\infty} \frac{1}{n(4n-1)} = \frac{6\ln 2 - \pi}{2}\,. \end{aligned} \tag{3}$$

These series are (absolutely) convergent by either the integral or series comparison test. We note that these identities were proved by the second author using combinations of the following (conditionally convergent) series, carefully treating the series as limits of combinations of Maclaurin series:[4]

$$\begin{aligned} 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots &= \ln 2\,; \\ 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots &= \frac{\pi}{4}\,. \end{aligned} \tag{4}$$

The second series above is known as the *Mādhava-Gregory-Leibniz series*.

Proof of Theorem 1. We compute $\frac{1}{2}(S_{15} - S_{16})$, with S_{15} and S_{16} as in (3) above. We note that $S_{15} - S_{16}$ is equal to $2(\pi/3 - \ln 2)$ and also to

$$\sum_{n=1}^{\infty} \frac{1}{n(4n-3)} - \sum_{n=1}^{\infty} \frac{1}{n(4n-1)} = \sum_{n=1}^{\infty} \frac{(4n-1)-(4n-3)}{n(4n-1)(4n-3)} = \sum_{n=1}^{\infty} \frac{2}{n(4n-1)(4n-3)}\,.$$

Dividing by two completes the proof. □

With a little arithmetic, the identities in (1) and Theorem 1 give new formulas for π:

$$\pi = 3\sum_{n=1}^{\infty} \frac{1}{n(2n-1)(4n-3)} = 3\ln 2 + 3\sum_{n=1}^{\infty} \frac{1}{n(4n-1)(4n-3)}\,. \tag{5}$$

3 Further formulas

With the identity in Theorem 1 now in hand, we set our sights on supplementing the list of identities given in [8] with further identities of similar forms. Theorem 1 is a low-hanging fruit that was directly asked for in the previous paper. Upon further inspection, we were able to spot other gaps in the list of twenty-seven identities.

The infinite series listed in [8] are classified according to the number of linear factors in the denominators of the summands. Observing that there were certain factors like $4n+1$ that showed up only rarely in the denominators of series in the list of identities in [8], we proved new identities of similar types.

[3] The identity for S_{15} is proved in [3] as the decagonal case of a formula related to figurate numbers.

[4] They are the limiting cases of $\ln(1+x)$ as $x \to 1^-$, and $\arctan x$ as $x \to 1$, respectively. The reader is referred to [1, 9] for details about Taylor series representations for π.

Theorem 2. *We have the following identities:*

(a)
$$\sum_{n=1}^{\infty} \frac{1}{n(2n-1)(4n-1)(4n-3)} = 2\ln 2 - \frac{\pi}{3}\,;$$

(b)
$$\sum_{n=1}^{\infty} \frac{1}{n(4n+1)(4n-1)(4n-3)} = \frac{\pi}{12} - \ln 2 + \frac{1}{2}\,;$$

(c)
$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(4n+1)(4n-1)(4n-3)} = \frac{1}{6}\left(\ln 2 - \frac{\pi}{4} + \frac{1}{2}\right);$$

(d)
$$\sum_{n=1}^{\infty} \frac{1}{n(2n-1)(4n+1)(4n-3)} = \frac{2}{3}\left(1 - \ln 2\right)\,;$$

(e)
$$\sum_{n=1}^{\infty} \frac{1}{n(2n-1)(4n+1)(4n-1)(4n-3)} = \frac{1}{3}\left(4\ln 2 - \frac{\pi}{2} - 1\right).$$

Proof. These identities are proved similarly to Theorem 1 above. Let S_1 denote the infinite series on the left-hand side of Theorem 1. Let S_a, S_b, S_c, etc., denote the series in identities (a), (b), (c), etc., of Theorem 2 above, respectively. Much like S_{15} and S_{16} defined in (3), let S_{17} and S_{19} denote the series in identities (17) and (19) of [8]:

$$S_{17} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)(4n-1)(4n-3)} = \frac{\ln 2}{2}\,;$$
$$S_{19} = \sum_{n=1}^{\infty} \frac{1}{(4n+1)(4n-1)(4n-3)} = \frac{\pi-2}{16}\,.$$

These are proved by linear combinations of identities (1), (3) and (4). By similar steps to the proof of Theorem 1, namely, combining series term-wise using fraction arithmetic, the identities in Theorem 2 follow from these formulas:

$$\begin{aligned}
S_a &= 2S_{17} - S_1\,;\\
S_b &= S_1 - 4S_{19}\,;\\
S_c &= \tfrac{1}{3}(S_{17} - 2S_{19})\,;\\
S_d &= 8S_c - S_a\,;\\
S_e &= 2S_c - S_b\,.
\end{aligned}$$

We note S_d and S_e both depend on preceding identities S_a and S_c in Theorem 2. As an example of one of these proofs, let us derive S_d explicitly. Note that $8S_c - S_a$ is equal to

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{8n}{n(2n-1)(4n+1)(4n-1)(4n-3)} &- \sum_{n=1}^{\infty} \frac{4n+1}{n(2n-1)(4n+1)(4n-1)(4n-3)}\\
&= \sum_{n=1}^{\infty} \frac{4n-1}{n(2n-1)(4n+1)(4n-1)(4n-3)} = S_d\,.
\end{aligned}$$

The same arithmetic steps, applied to the right-hand sides of (a) and (c), lead to (d). $\square$

4 Conclusion

The identities we prove above do not appear in standard tables of infinite series at our disposal [5, 6], nor in popular books about π such as [1, 2], nor among the well-known infinite series identities of Euler (see e.g. [4, 7]).

We note that identities such as these will not give approximations to π that converge rapidly enough to be useful for computations, by comparison to formulas used to compute billions upon billions of digits of π (see [11]), but they do converge reasonably quickly – generally more so in a given series, the greater the number of factors in the denominator. For example, S_{19} can be used to compute π:

$$\pi = 2 + 16S_{19}\,. \tag{6}$$

By summing just the first nine terms of S_{19}, Equation (6) yields the approximation $\pi \approx 3.140133\ldots$ accurate to two decimal places, and the first 217 terms of S_{19} gives $\pi \approx 3.141590\ldots$ accurate to five places. By contrast, to use the Mādhava-Gregory-Leibniz series $\pi = 4\,(1 - 1/3 + 1/5 - 1/7 + \cdots)$ in (4) to produce an estimate accurate to five decimal places takes $500{,}000$ terms [10]. On the other hand, Ramanujan proved an efficient formula whose $n = 1$ summand alone gives the first seven digits of π, with each additional term of the summation giving eight additional digits of the decimal expansion [12].[5]

We encourage the reader to seek further identities to supplement the lists of infinite series formulas made in [8] and in this paper, perhaps combining these formulas with known identities from the literature. Excellent starting points would be Dunham's survey [4] of the infinite series methods of Euler, and Beckmann's historical survey [1] of formulas for π. For further reading, see [3] in which a theory is developed of identities similar to those in [8] and in this work.

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[5]For milestones in computing digits of π, see [12].

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