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New geometrical and mechanical proofs of the Pythagorean Theorem

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1 Introduction

Being one of the oldest and certainly one of the most well-known theorems in mathematics, the Pythagorean Theorem has a large number of proofs. This paper presents two new proofs, one geometrical and one mechanical, of this theorem.

The first proof is inspired by Euclid's proof (see Proof `#1` from [1]), in which the squares of the side lengths of the right-angled triangle are interpreted as areas of squares sitting upon those sides. However, Euclid's proof can be intimidating in terms of the additional constructions involved, and might not be intuitive at first glance. The first proof uses Euclid's initial construction of the three squares but employs a more intuitive set of additional constructions which make the relevant congruences and reasoning feel much more motivated.

The second proof is purely mechanical and follows from reasoning about a physical system of an inclined rod sliding down a wall. Mechanical proofs of famous mathematical results are covered plentifully in Mark Levi's book *The Mathematical Mechanic* [2], which even contains a chapter dedicated to mechanical proofs of the Pythagorean Theorem.

Both of the proofs presented in this paper are, as far as I can tell, new and original, and do not appear in the hundreds of popular proofs online and in other literature, including the ones mentioned above.

2 Geometrical Proof

2.1 Construction

As in Figure 1, let us construct $\triangle ABC$ with a right angle at B. Then, construct squares $ABDE$, $BCFG$, and $ACHI$ on sides AB, BC, and AC, respectively. Let the extension of ray IA intersect DE at L and meet with the extension of ray CB at J. Similarly, let the extension of ray HC intersect GB at N and meet with the extension of ray FG at K. Construct line LK and let its intersection with the extension of ray CB be M. Finally, construct the perpendicular KP from K to the extension of ray CB.

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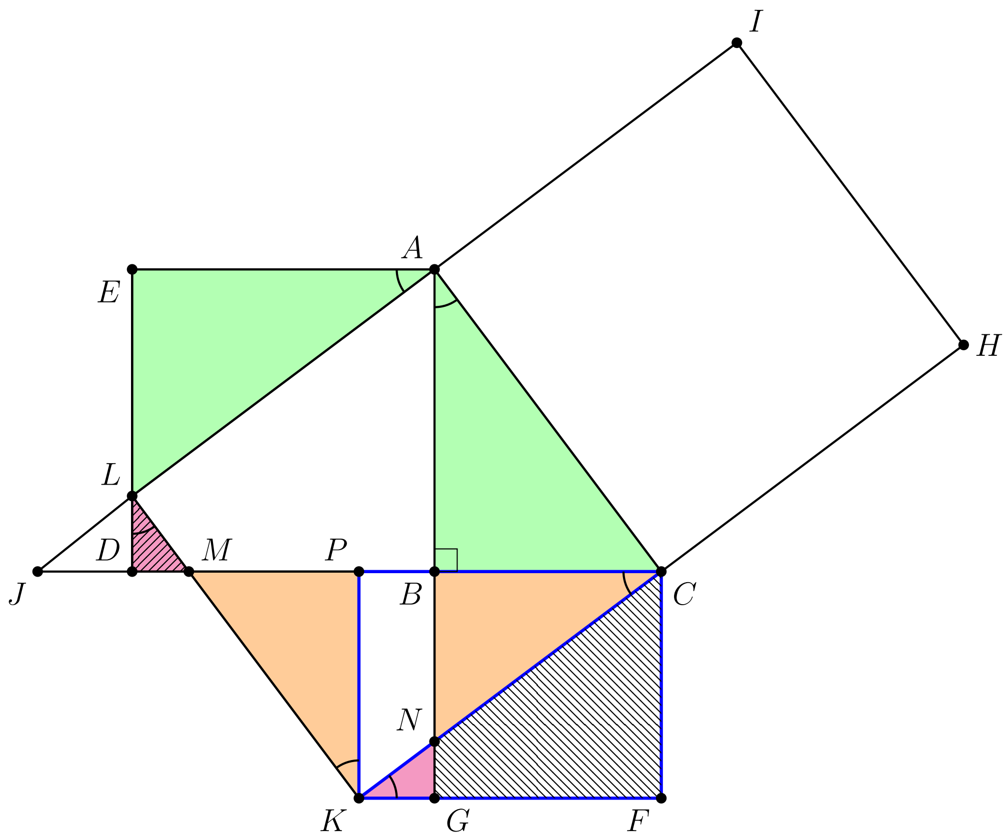

Figure 1: Geometrical Construction

2.2 The proof

To prove the Pythagorean Theorem, it suffices to prove that the area of square $ACHI$ is the sum of the areas of the squares $ABDE$ and $BCFG$.

We first show that the squares $ACHI$ and $ACKL$ are congruent. Then, it will simply suffice to show that the area of square $ACKL$ is equal to that of squares $ABDE$ and $BCFG$. The advantage then will be that $ACKL$ shares large "chunks" of areas (namely, $ABML$ and BNC) with $ABDE$ and $BCFG$.

Since we are dealing with right-angled triangles, triangle congruences will be shown by proving that a side and one of the angles apart from the right angle are equal. For brevity, we complete the necessary angle chasing first and mark all the equal angles in Figure 1:

$$\angle EAL = 90^\circ - \angle LAB = \angle BAC$$
$$\angle BAC = 90^\circ - \angle BCA = \angle PCK$$
$$\angle PCK = \angle CKF \quad \text{(since } CP \parallel FK\text{)}$$

The construction described at the beginning yields lots of congruent triangles, so we can now begin with the following lemmas:

Lemma 1. *$\triangle ABC \cong \triangle AEL \cong \triangle CPK \cong \triangle KFC$.*

Proof. $\triangle ABC \cong \triangle AEL$ since, by construction, $|AB| = |AE|$ and $\angle EAL = \angle BAC$. $\triangle ABC \cong \triangle CPK$ since, by construction, $|PK| = |BG| = |BC|$ and $\angle BAC = \angle PCK$. $\triangle CPK \cong \triangle KFC$ since $KPCF$ forms a rectangle, and diagonal KC splits it into two congruent triangles. □

Lemma 2. *$ACKL$ is a square. In particular, since AC is a common side of both squares $ACKL$ and $ACHI$, they are congruent to each other.*

Proof. By construction, $\angle LAC = \angle ACK = 90°$. By Lemma 1, $|AC| = |AL| = |CK|$. Therefore, $ACKL$ is a square with side length $|AC|$. Thus, $ACKL$ is a square congruent to $ACHI$. □

Since $ACKL$ has been shown to be a square and since $\angle CKM = 90°$, we can continue our angle chasing:

$$\angle CKF = 90° - \angle PKC = \angle PKM = \angle DLM\,.$$

This concludes all the angle markings in Figure 1.

Lemma 3. *$\triangle LDM \cong \triangle KGN$.*

Proof. By Lemma 1, $|LD| = |ED| - |EL| = |AB| - |BC| = |FK| - |FG| = |KG|$. As marked in Figure 1 from prior angle chasing, $\angle DLM = \angle GKN$. □

Lemma 4. *$\triangle NBC \cong \triangle MPK$.*

Proof. As in Figure 1 from prior angle chasing, $\angle PKM = \angle BCN$. By construction, $|BC| = |BG| = |PK|$. □

In Lemma 2, we have already proven our claim that squares $ACHI$ and $ACKL$ are congruent. The parts shaded with same colours in Figure 1 are congruent, as proven in the preceding lemmas, and thus equal in areas.

As mentioned previously, $ABML$ and BNC are common areas of square $ACKL$ with the smaller squares. Also, the area ABC in square $ACKL$ is the same as the area AEL in the smaller square $ABDE$. So, we just need to prove that the non-overlapping and non-equal areas of the smaller squares, which are LDM and $NGFC$ (line-shaded in Figure 1), sum to the non-overlapping and non-equal area of square $ACKL$, which is $BNKM$. Observe that:

$$\begin{aligned}
[LDM] + [NGFC] &= [KGN] + [NGFC] \quad \text{(since } \triangle LDM \cong \triangle KGN \text{ by Lemma 3)} \\
&= [KFC] \\
&= [CPK] \quad \text{(since } \triangle KFC \cong \triangle CPK \text{ by Lemma 1)} \\
&= [NBC] + [BNKP] \\
&= [MPK] + [BNKP] \quad \text{(since } \triangle NBC \cong \triangle MPK \text{ by Lemma 4)} \\
&= [BNKM]\,.
\end{aligned}$$

This concludes the proof. □

3 Mechanical Proof

3.1 Physical Setup

As in Figure 2, let us consider a rigid rod of length L initially resting against a wall (see left picture). Then, it is given a tiny nudge to the right at its lowest point. As a result, it starts sliding down the wall and to the right. At some time during its descent, the rod makes an angle θ with the floor (see right picture). All surfaces are frictionless.

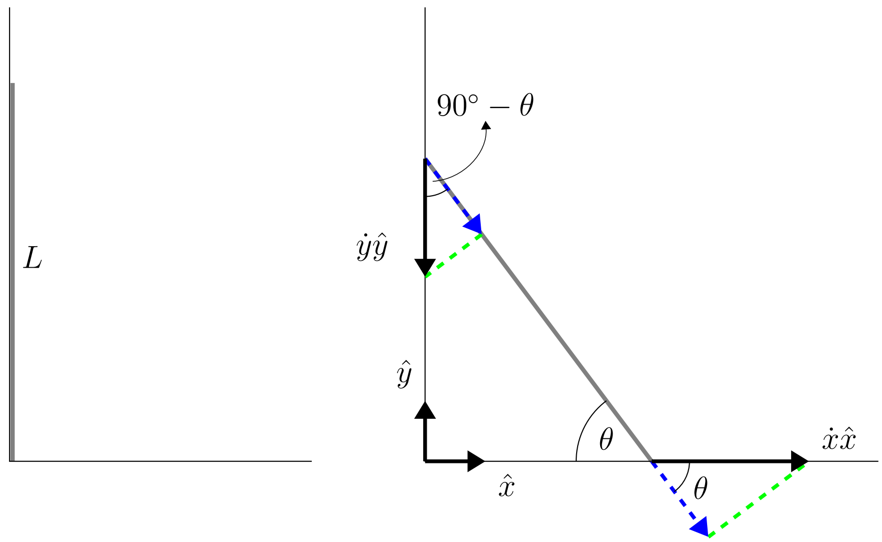

Figure 2: Initial configuration (left) and configuration at a general angle θ (right)

3.2 The proof

The top-most point of the rod moves vertically down with velocity $-\dot{y}\hat{y}$, where the dot represents the derivative with respect to time. We can decompose the vector $\dot{y}$ into two components, along and perpendicular to the rod. Similarly, the bottom-most point of the rod moves horizontally to the right with velocity $\dot{x}\hat{x}$, which can also be decomposed into two components along and perpendicular to the rod.

Here is the observation that immediately leads to a proof of the Pythagorean Theorem: since the rod is rigid, the velocities (the blue dotted vectors) of the top-most and bottom-most points on the rod must be equal. That is,

$$\dot{x}\cos(\theta) = -\dot{y}\cos(90° - \theta) = -\dot{y}\sin(\theta)\,;$$

or, in other words,

$$\frac{dx}{dt}\cos(\theta) = -\frac{dy}{dt}\sin(\theta)\,. \tag{1}$$

Note the negative sign before $\dot{y}$. It is there because the rate of change of y, unlike that of x, is negative, since the y coordinate of the top of the rod decreases with time.

We solve the differential equation (1) by using separation of variables:

$$dx = -dy\ \tan(\theta) = -dy\,\frac{y}{x}\,,$$

so

$$x\,dx = -y\,dy\,.$$

To integrate this equation, note that x increases from 0 to x and y decreases from L to y:

$$\int_0^x x\,dx = -\int_L^y y\,dy\,,$$

so $\frac{x^2}{2} = -\left(\frac{y^2}{2} - \frac{L^2}{2}\right)$. Therefore,

$$x^2 + y^2 = L^2\,.$$

Since we could have started with a rod of any length, we have proved the Pythagorean Theorem. □

4 Conclusion and Reflections

In the first proof, the congruences of the many triangles and more importantly, of the squares $ACKL$ and $ACHI$, were mostly inspired by the diagram. The fact that square $ACKL$ had many overlaps with the two smaller squares made the subsequent demonstration, coupled with the previously proven congruences between the different triangles, quite simple.

The second proof was much more direct once the velocities of the two ends along the rod were equated. The novelty of this proof is due to the physical setup and the realization that the rigidity condition corresponds to velocity equality along the rod.

References

[1] Cut The Knot, *Pythagorean Theorem*, https://www.cut-the-knot.org/pythagoras/, last accessed on 2025-02-26.

[2] M. Levi, *The Mathematical Mechanic: Using Physical Reasoning to Solve Problems*, Princeton University Press, 2012.