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On convex kinematics: a resurrection of Holditch's Theorem

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1 Context

Holditch's Theorem [2] was published in 1857 by Reverend Hamnet Holditch in *The Quarterly Journal of Pure and Applied Mathematics*. It describes a geometric invariant, in the form of an area, for any convex region in $\mathbb{R}^2$. The original proof was only ten lines long, relying on the construction of two points on a rotating chord, and comparing the area bounded by each over a full rotation of 2π radians. Though admirably concise, the proof hides much of the intuition involved in the theorem, relying on clever, yet unobvious constructions and manipulations. For that reason, this paper seeks proof of Holditch's Theorem in an intuitive sense relying upon the concrete geometry of Green's and Stewart's theorems.

2 Definitions and theorems

Definition 1 (Line integrals)**.** *Line integrals are a type of integral used to measure some function defined on points along a curve in $\mathbb{R}^n$. More precisely, for $f(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ with a curve $C \in \mathbb{R}^n$, the associated line integral is defined as follows:*

$$\int_c f(x_1, x_2, \ldots, x_n)ds = \lim_{|\Delta|\to 0} \sum_i f(x_{1i}, x_{2i}, \ldots, x_{ni})\Delta S_i \,,$$

where $|\Delta|$ is the greatest length of any subinterval of any partition of C.

Theorem 2. (Green's Theorem [1]) *For a positively oriented, piecewise smooth, simple closed curve $C \subset \mathbb{R}^2$ and a region D bounded by C, with a 2-dimensional vector field $\vec{F}$ of x and y components, P and Q respectively, and a counter-clockwise integration along C,*

$$\oint_C (P\,dx + Q\,dy) = \iint_D \left(\nabla \times \vec{F}\right) dA \,.$$

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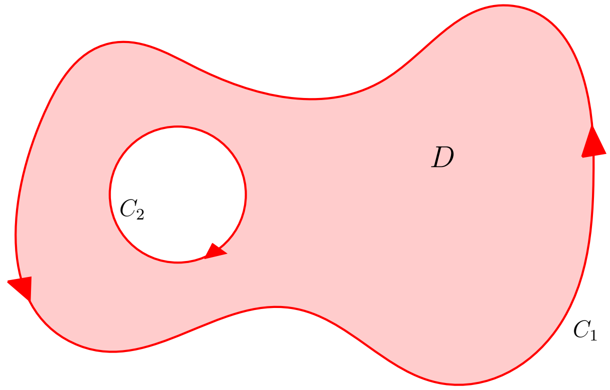

Figure 1: Green's Theorem

Corollary 3 (Area of a region in $\mathbb{R}^2$)**.** *For any region $D \subset \mathbb{R}^2$ bounded by piecewise smooth, simple closed curve C,*

$$\frac{1}{2} \oint_{\partial c} xdy - ydx = [D] \,,$$

where $[D]$ is the area enclosed by the curve C.

Proof. Let $\vec{F} = \langle -y, x \rangle$. By Theorem 2,

$$\oint_{\partial c} xdy - ydx = \iint_D \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} \right) dA = 2 \iint_D dA \,.$$

By the definition of a double integral, the right side is simply double the area of the region D. So, dividing by 2,

$$\frac{1}{2} \oint_{\partial c} xdy - ydx = [D] \,. \qquad \square$$

Theorem 4. (Stewart's Theorem [4]) *Let $\triangle ABC$ be a triangle and let D be a point on the line segment $\overline{BC}$. Then*

$$|\overline{AB}|^2 |\overline{CD}| + |\overline{AC}|^2 |\overline{BD}| = |\overline{AD}|^2 |\overline{BC}| + |\overline{BD}| \, |\overline{DC}| \, |\overline{CB}| \,.$$

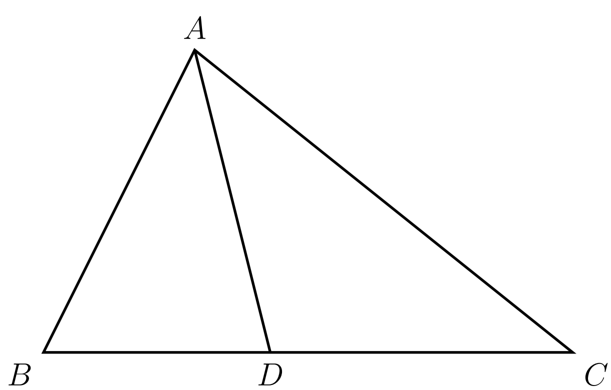

Figure 2: Stewart's Theorem

3 Holditch's Theorem

Theorem 5. (Holditch's Theorem [2]) *Let Ω be a convex region in $\mathbb{R}^2$ and let L be a line segment of length ℓ that connects points on the boundary of Ω. As we move one end of L around the boundary maintaining the length ℓ, the other end will also move about this boundary, and some point P on L that splits the line into segments of lengths p and q, respectively, will trace out a curve within Ω that bounds a (smaller) region Γ. For all convex regions Ω, $[\Omega] - \pi pq = [\Gamma]$, where $[\Omega]$ and $[\Gamma]$ are the areas bounded by the curves Ω and Γ, respectively.*

We must note that not all lengths ℓ for the line segment L will allow L to travel continuously about the border of a convex region in $\mathbb{R}^2$. So, for the duration of the proof, we will assume that we have chosen some ℓ that allows L to travel continuously about the border of the region Ω.

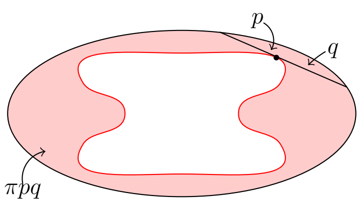

Figure 3: Holditch Curve - Ellipse

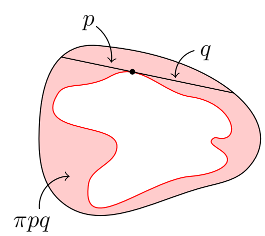

Figure 4: Holditch Curve - Arbitrary Region

Lemma 6. *For a Holditch curve C, parametrised by $\langle k\cos\theta, k\sin\theta\rangle$,*

$$\frac{1}{2}\oint_{\partial C} k^2 d\theta = [C]\,,$$

where $[C]$ is the area bounded by C.

Proof. Let Γ be parametrised by $\langle k\cos\theta, k\sin\theta\rangle$ where k is a function of θ. We pick such a parametrization such that $\theta \in [0, 2\pi]$, to maintain the idea of completing a "full rotation". Now we calculate the differentials of our parametrization and get

$$\begin{aligned} dx &= -k\sin\theta d\theta + k'\cos\theta d\theta\,; \\ dy &= \quad k\cos\theta d\theta + k'\sin\theta d\theta\,. \end{aligned}$$

By Corollary 3,

$$\frac{1}{2}\oint_{\partial\Gamma} (k\cos\theta)(k\cos\theta + k'\sin\theta)d\theta - (k\sin\theta)(-k\sin\theta + k'\cos\theta)d\theta = [\Gamma]\,.$$

By expanding and simplifying, we have

$$\frac{1}{2}\oint_{\partial\Gamma} k^2 d\theta = [\Gamma]\,. \qquad \square$$

Noting that k is a function of θ and we are integrating with respect to θ, we recall our parametrization as $\theta \in [0, 2\pi]$. To get a geometric understanding of what this represents, we construct a diagram of the situation (in an arbitrary manner), where the outer region is Ω and the inner is Γ. We now pick some point s in the region Γ as the origin for our parametrization for Γ. We know that k is the distance from s to $\partial\Gamma$ as a function of θ, and our line is split into two, not necessarily equal, segments p and q by the line of length k. We construct the distance from our arbitrary point to the two points in which L intersects $\partial\Omega$ and denote those distances by r_1 and r_2.

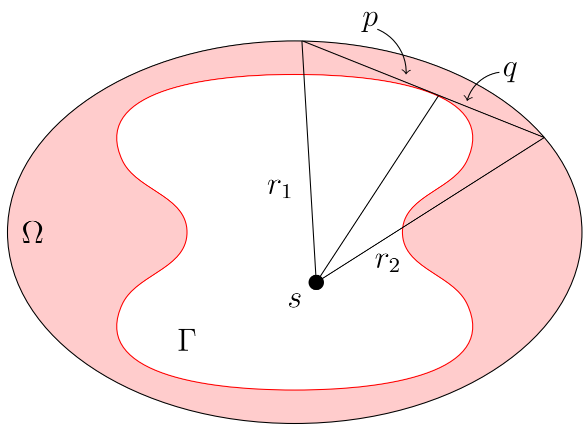

Figure 5: Holditch Curve - Triangle

We can now prove Holditch's Theorem.

Proof of Theorem 5. First, we note that some form of this triangle (dynamically changing) will exist for all convex regions trivially, but will clearly not exist for concave regions; this explains why the proof only holds true for convex regions. By applying Theorem 4, we see that

$$k^2 = \frac{pr_2^2 + qr_1^2}{p+q} - pq\,.$$

Thus by Lemma 6,

$$[\Gamma] = \frac{1}{2}\int_0^{2\pi}\left(\frac{pr_2^2 + qr_1^2}{p+q} - pq\right)d\theta = \frac{1}{2}\left(\int_0^{2\pi}\frac{pr_2^2}{p+q}d\theta + \int_0^{2\pi}\frac{qr_1^2}{p+q}d\theta - \int_0^{2\pi} pq\,d\theta\right).$$

Removing the constant $\frac{p}{p+q}$, the first integral evaluates to $[\Omega]$ as follows:

$$[\Omega] = \int\int_\Omega dxdy = \int_0^{2\pi}\int_0^R rdrd\theta = \int_0^{2\pi}\frac{r_2^2}{2}\,d\theta\,,$$

and likewise for the second integral. So, returning the constants, we are left with

$$[\Gamma] = \frac{1}{2}\left(\frac{2p[\Omega]}{p+q} + \frac{2q[\Omega]}{p+q} - \int_0^{2\pi} pqd\theta\right).$$

By evaluating the integral and then distributing the $\frac{1}{2}$, we arrive at

$$[\Omega] - \pi pq = [\Gamma]\,,$$

which confirms is what Theorem 5 asserts. $\square$

4 Conclusion

In this article, we presented a novel proof for an old theorem that relates the area between a convex region, and an inner region created by a point on a line, swept around the border. It relies on fundamental ideas in geometry and calculus, namely Stewart's and Green's Theorems. This provides a more intuitive approach to the previously abstract invariant. Furthermore, in contrast with the initial proof, this article uses mechanisms extendable beyond planar geometry, leaving the door open for future developments in higher-dimensional spaces. For more information regarding Holditch's Theorem and intuitive approaches to explaining it, see [3].

Acknowledgements

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References

[1] Wikipedia, *Green's Theorem*, https://en.wikipedia.org/wiki/Green's_theorem, last accessed on 2025-08-19.

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[4] M. Stewart, *Some General Theorems of Considerable Use in the Higher Parts of Mathematics*, Sands, Murray and Cochran, Edinburgh, 1746.