**Parabola Volume 61 Issue 2 (2025), Article 6**

# A neat modification of the integration by parts formula

**David Michael Indraputra**[1] and **Benny Yong**[2]

## Introduction

Calculus is everywhere. We often face integral and calculus-related problems when dealing with Physics, Astronomy, Statistics and other subjects that require quite advanced mathematics. This is the reason why calculus is added to the high school mathematics curriculum in almost all countries in the world.

$$\int x^{2022}\,\mathrm{d}x \qquad \int \sin x\,\mathrm{d}x \qquad \int (x^3+2)^7 x^5\,\mathrm{d}x$$

The above examples are some integrals that can be easily solved using only algebra or trigonometric identities. Not all integrals are easy to solve; for instance, see the following integrals:

$$\int \frac{x^2-1}{(x^2+1)\sqrt{x^4+1}}\,\mathrm{d}x \qquad \int \frac{x^2}{(x\sin x+\cos x)^2}\,\mathrm{d}x \qquad \int_{-\infty}^{\infty} \frac{1}{(e^x-x+1)^2+\pi^2}\,\mathrm{d}x$$

Some integrals may be really hard and brutal to solve without a certain technique or trick. There are plenty of well-known integration techniques: e.g., $u$-substitution, integration by parts, reverse chain rule, partial fraction expansion, trigonometric substitution and Weierstrass substitution. If you are in an exam and desperate to solve integration problems, then you may also expand the integrand by using Taylor series expansion and write your final answer in infinite summations (though I don't personally recommend this). You see that there are so many integration techniques you have to learn, which may confuse you about picking the right technique to solve a problem. This may be the reason why there are a lot of students who have been struggling in calculus classes.

In the present paper, we will only be looking at the integration by parts technique, which plays a major role in high school mathematics and in basic calculus. The integration by parts formula is essentially just integrating by way of the chain rule formula. Indeed, since

$$(uv)' = u'v + uv'\,,$$

[1]David Michael Indraputra is a student who is pursuing a major in Electrical Engineering and minors in AI and Mathematics at the National University of Singapore. (davidmi07@u.nus.edu)

[2]Benny Yong is an Associate Professor at the Centre for Mathematics and Society, Department of Mathematics, Parahyangan Catholic University, Bandung, Indonesia. (benny_y@unpar.ac.id)

it follows that

$$\int u \,\mathrm{d}v = uv - \int v \,\mathrm{d}u \,.$$

In basic calculus, the only thing you need to practice is choosing the right functions $u$ and $v$. Based on the advice give in [2], the two criteria for choosing $u$ and $v$ are that $v$ should be easy to find from $\mathrm{d}v$, and that the integral $\int v \,\mathrm{d}u$ should be better or easier to solve than $\int u \,\mathrm{d}v$. In order to satisfy both criteria, you may remember the word LIATE that stands for Logarithmic Inverse Algebraic Trigonometric Exponential. This abbreviation can help you in selecting the right $u$ and $\mathrm{d}v$.

If you have already decided the functions $u$ and $v$, then you may use the tabular or D-I method (see [1]) to help you in calculations. The table consists of two rows with your chosen differentiated and integrated functions being separated on the two rows. The first row has the sign of positive, the second negative, the third positive and so on. However, there are some problems which may take much time to do using the regular way we have mentioned above. For instance, suppose that you are given a task to solve $\int x \arctan x \,\mathrm{d}x$. You can choose $\arctan x$ as $u$ and $x \,\mathrm{d}x$ as $\mathrm{d}v$. It will, however, exhaust your time because you have to solve another unnecessary integral. In the present paper, we will discuss a clever integration by parts trick which can make calculations much easier and save your life.

# The trick

We start from the well-known integration by parts formula, which is given by

$$\int f(x)g'(x) \,\mathrm{d}x = f(x)g(x) - \int f'(x)g(x) \,\mathrm{d}x \,.$$

Notice that, if we add a constant $k$ inside a derivative, then the derivation of the function remains the same. Thus, by transforming $g(x) \to g(x) + k$, the integration by parts formula is now

$$\int f(x)g'(x) \,\mathrm{d}x = \int f(x)\big(g(x)+k\big)' \,\mathrm{d}x = f(x)\big(g(x)+k\big) - \int f'(x)\big(g(x)+k\big) \,\mathrm{d}x \,.$$

You can choose any constant $k$ such that the integral on the right-hand side becomes much easier to solve! Note that this trick is really helpful if you can cancel out $g(x) + k$ with something. We can generalize this idea for integration by parts involving the product of many functions. Let's say that we have functions $f_i(x), \ldots, f_n(x)$ that are differentiable on the interval $[a, b]$. By making use of the product rule, we see that

$$\left(\prod_{i=1}^{n} f_i(x)\right)' = \sum_{j=1}^{n} f_j'(x) \prod_{i\neq j}^{n} f_i(x) \,.$$

Integrate both sides with respect to $x$ and re-arrange the terms to get

$$\int_a^b \left(\prod_{i=1}^{n-1} f_i(x)\right) f_n'(x) \,\mathrm{d}x = \left(f_n(x) \prod_{i=1}^{n-1} f_i(x)\right)\Bigg|_a^b - \sum_{j=1}^{n-1} \int_a^b f_n(x) \left(\prod_{i\neq j}^{n-1} f_i(x)\right) f_j'(x) \,\mathrm{d}x \,.$$

We can do similar transformation like before, namely to change $f_n(x) \to f_n(x) + k$ without harming the integral. Let us also rename $f_n(x)$ as $g(x)$:

$$\int_a^b \left(\prod_{i=1}^{n-1} f_i(x)\right) g'(x)\,\mathrm{d}x = \left(\big(g(x)+k\big)\prod_{i=1}^{n-1} f_i(x)\right)\Bigg|_a^b - \sum_{j=1}^{n-1}\int_a^b \big(g(x)+k\big)\left(\prod_{i\neq j}^{n-1} f_i(x)\right) f_j'(x)\,\mathrm{d}x.$$

For multi-variable functions such as $\mathbf{F}(x_1,\ldots,x_n) = f(x_1,\ldots,x_n)\mathbf{g}(x_1,\ldots x_n)$, our trick also applies when $f$ is a scalar-valued function and g is a vector-valued function on $\Omega$, an open bounded subset of $\mathbb{R}^n$, with a piecewise smooth boundary $\Gamma = \partial\Omega$.

To see how the trick applies to this generalisation, the product rule for divergence states that

$$\nabla\cdot(f\mathbf{g}) = f\nabla\cdot\mathbf{g} + \mathbf{g}\cdot\nabla f\,.$$

By the Divergence Theorem [3],

$$\int_\Omega \nabla\cdot\mathbf{F}\,d\Omega = \oint_\Gamma \mathbf{F}\cdot\mathbf{n}\,d\Gamma\,.$$

By substituting $F = f\mathbf{g}$ and applying the product rule, we have

$$\int_\Omega f\nabla\cdot\mathbf{g}\,d\Omega = \oint_\Gamma f\mathbf{g}\cdot\mathbf{n}\,d\Gamma - \int_\Omega \mathbf{g}\cdot\nabla f\,d\Omega\,.$$

Now, replacing $\mathbf{g}\to\mathbf{g}+\mathbf{k}$ with constant vector $\mathbf{k}$, we have

$$\int_\Omega f\nabla\cdot\mathbf{g}\,d\Omega = \oint_\Gamma f(\mathbf{g}+\mathbf{k})\cdot\mathbf{n}\,d\Gamma - \int_\Omega (\mathbf{g}+\mathbf{k})\cdot\nabla f\,d\Omega\,.$$

Now we are going to apply this trick to solve some integration problems.

**Example 1.** Calculate $\displaystyle\int \arctan\sqrt{x+3}\,\mathrm{d}x$.

To solve this problem, one may use $u$-substitution $x+3 = u^2$ and $\mathrm{d}x = 2u\,\mathrm{d}u$. However, we can apply the trick to solve this problem much quicker. Here, we have that $g(x) = x$ and $f(x) = \arctan\sqrt{x+3}$:

$$\int \arctan\sqrt{x+3}\,\mathrm{d}x = (x+k)\arctan\sqrt{x+3} - \int (x+k)\frac{1}{x+4}\left(\frac{1}{2\sqrt{x+3}}\right)\mathrm{d}x\,.$$

Now choose $k = 4$ to cancel out the $x+4$ term on the denominator. Integrate the remaining expression and we will achieve

$$\int \arctan\sqrt{x+3}\,\mathrm{d}x = (x+4)\arctan\sqrt{x+3} - \sqrt{x+3} + C\,.$$

You will get the same answer by solving the same problem with $u$-substitution and regular integration by parts method. It also works for integrals involving trigonometry, just as shown in the following example.

**Example 2.** Compute $\int \sin(2x)\arctan(\sin x)\,\mathrm{d}x$.

Since $\left(\sin^2 x\right)' = \left(\frac{1}{2}(1-\cos(2x))\right)' = \sin(2x)$, we can apply the modified integration by parts formula as follows:

$$\begin{aligned}\int \sin(2x)\arctan(\sin x)\,\mathrm{d}x &= \int (\sin^2 x)' \arctan(\sin x)\,\mathrm{d}x \\ &= (k+\sin^2 x)\arctan(\sin x) - \int (k+\sin^2 x)\,\frac{\cos x}{1+\sin^2 x}\,\mathrm{d}x\,.\end{aligned}$$

You may choose $k=1$ so that the terms $1+\sin^2 x$ and $k+\sin^2 x$ cancel each other. The integration then become much easier to solve than by using the substitution $u=\sin x$ and solving it the regular way. The integral is, therefore, equal to

$$\int \arctan(\sin x)\,(\sin^2 x)'\,\mathrm{d}x = (1+\sin^2 x)\arctan(\sin x) - \sin x + C\,.$$

For some problems, this trick may only take one step to finish the integration. Now we shall solve a more challenging problem.

**Example 3.** Calculate $I = \int \left(1 - \frac{1}{x+1} + \frac{x}{(x+1)^2}\right)\ln\left(x^3-1\right)\mathrm{d}x$ where $x>1$.

We can rewrite the terms in bracket as $\frac{x(x+2)}{(x+1)^2}$ which is the derivative of $g(x) = \frac{x^2}{x+1}$. Apply our trick to the integral with $f(x) = \ln(x^3-1)$ and $g(x) = \frac{x^2}{x+1}$:

$$\int\!\left(1-\frac{1}{x+1}+\frac{x}{(x+1)^2}\right)\ln\left(x^3-1\right)\mathrm{d}x = \left(\frac{x^2}{x+1}+k\right)\ln(x^3-1) - \int\!\left(\frac{x^2}{x+1}+k\right)\frac{3x^2}{x^3-1}\,\mathrm{d}x.$$

Choose $k=1$, so that $g(x)+k = \frac{x^2+x+1}{x+1} = \frac{x^3-1}{x^2-1}$:

$$I = \left(\frac{x^2}{x+1}+1\right)\ln(x^3-1) - \int \frac{3x^2}{x^2-1}\,\mathrm{d}x = \frac{x^3-1}{x^2-1}\ln(x^3-1) - 3\int\left(1+\frac{1}{x^2-1}\right)\mathrm{d}x\,.$$

We can split $\frac{1}{x^2-1}$ into $\frac{1}{2}\left(\frac{1}{x-1}-\frac{1}{x+1}\right)$. Thus, we see that the monstrous integral is equal to

$$I = \frac{x^3-1}{x^2-1}\ln(x^3-1) - 3x + \frac{3}{2}\ln\left(\frac{x+1}{x-1}\right) + C\,.$$

If we do not use the trick, the integral is relatively complicated to solve. To solve it, you may make use of the partial fraction decomposition method.

To challenge your understanding, you may try solving the following problems.

**Exercise 1.** Show that the function $y(x) = (x(2x^2-3x+3)-1)\arctan^2(x^2-x+1)$ is antisymmetric about the line $x=1/2$ and sketch the graph. Hence, by calculating the integral directly show that

$$\int_0^1 (x(2x^2-3x+3)-1)\arctan^2(x^2-x+1)\,\mathrm{d}x = 0\,.$$

**Hint:** Try factoring $x(2x^2 - 3x + 3) - 1$ into something "nice".

**Exercise 2.** In this problem, you are asked to calculate three non-related integrals.

(a) Using partial fractions, evaluate

$$A_1 = \int \frac{\ln x}{(x-1)^5}\,\mathrm{d}x\,.$$

(b) Using another partial fractions, evaluate

$$A_2 = \int \frac{\ln(x^2 - 3x + 3)}{(x-1)^5}\,\mathrm{d}x\,.$$

(c) Using yet another partial fractions, evaluate

$$A_3 = \int \frac{2-x}{(x-1)^4(x^3 - 3x + 3)}\,\mathrm{d}x\,.$$

(d) Compute the function $f(x) = 4A_1 + 4A_2 + 3A_3$.

(e) There is another way to compute $f(x)$. Compute the following function using the modified integration by parts formula.

$$g(k,x) = \int \frac{\ln(k^3 + (x-k)^3)}{(x-k)^4}\,\mathrm{d}x\,.$$

(f) Show that our function $f(x)$ is related to $g(k,x)$ by $f(x) = \frac{\partial g}{\partial k}$ on $k = 1$. Hence, show that the result that you get for part (d) is equivalent to the result we get for this part. This method is called the *Feynman technique of integration*.

**Exercise 3.** In quantum mechanics, we always encounter a wave function as a mathematical description of the quantum state of an isolated system. Based on the known postulates of quantum physics, the wave function of a quantum mechanical system $\Psi(\mathbf{r},t)$ depends on the coordinates of the particle, $\mathbf{r}$, and the time, $t$. The amplitude square of the wave function itself $|\Psi(\mathbf{r},t)|^2$ denotes the probability density of finding the particle in the volume element $\mathrm{dVol}$ located at $\mathbf{r}$ and time $t$. This means that the wave function must satisfy the normalisation condition

$$\int_{\text{all volume}} |\Psi(\mathbf{r},t)|^2\,\mathrm{dVol} = \int_{\text{all volume}} \Psi^*(\mathbf{r},t)\Psi(\mathbf{r},t)\,\mathrm{dVol} = 1\,,$$

where $\Psi^*$ denotes the complex conjugate of $\Psi$. Since the wave function is normalised, the average value of a variable $V$ with operator $\hat{V}$ is equal to

$$\langle V \rangle = \int_{\text{all volume}} \Psi^*(\mathbf{r},t)\hat{V}\Psi(\mathbf{r},t)\,\mathrm{dVol}$$

with the operator $\hat{V}$ being squeezed between $\Psi^*$ and $\Psi$. Another fact is that when we take an operator on a linear wave function, we must get the wave function back and, hence, there must exist an eigenvalue of the operator. Recall that the Hamiltonian of a system is $H = \frac{p^2}{2m} + V$. Thus, by taking the Hamiltonian operator on $\Psi$, we must get the total energy of the system $E$; this is the general form of Schrödinger's equation $\hat{H}\Psi = E\Psi$. It is known that the momentum operator is defined as $\hat{p} = -i\hbar\nabla$ in its 3-dimensional form.

Consider a (so-called) strange particle with mass $m$ and a time-independent 1-dimensional wave function $\Psi(x) = A\arctan(kx)$ where $k$ is the "wavelength" of the particle. This wave function is $0$ outside of the interval $-1 \leq kx \leq 1$.

(a) Find the normalization constant $A$. Express your answer in terms of $k$ and the Catalan constant

$$C = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \approx 0.916\,.$$

(b) Show that the average value of position $\langle x \rangle$ is zero. Deduce also the value of $\langle x^2 \rangle$.

(c) Show that the average value of momentum $\langle p \rangle$ is zero.

(d) Define the average potential energy of the system $\langle V \rangle$ to be equal to zero between the region $-1 \leq kx \leq 1$, and infinite outside of it (since no particle is able to escape the potential well). Find the value of $\langle p^2 \rangle$ and, hence, find the average total energy of the particle $\langle E \rangle$.

(e) If the wave function $\Psi$ is a Gaussian-like function, then the Heisenberg uncertainty principle states that $\Delta p \Delta x \geq \hbar/2$. Find the "Heisenberg uncertainty principle" for our wave function. How is the value you get compared to $\hbar/2$? Is it greater than, less than, or equal to?

# References


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