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Characterisation of an $\mathbb{R}^2$ tight frame: process and extension

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1 Introduction

Vectors in $\mathbb{R}^2$ can nicely be described as directed line segments in a plane. We also do not distinguish the position of these line segments in the plane. Hence, any vector in $\mathbb{R}^2$ can be represented by an ordered pair of real numbers, which we shall write as a column matrix. Along with elementary vector operations, this interpretation of $\mathbb{R}^2$ allows us to make basic geometric observations that are familiar even for a high school student.

A *basis* is a collection of vectors in a vector space ($\mathbb{R}^2$ for our purposes) that can be used to represent any other vectors. An example of a basis would be the vector collection $\mathbf{e}_1 := [1\ 0]^{\mathbf{T}}$ and $\mathbf{e}_2 := [0\ 1]^{\mathbf{T}}$ for the vector space $\mathbb{R}^2$. From these two vectors, we have the following for any vector $\mathbf{x} = [a\ b]^{\mathbf{T}} \in \mathbb{R}^2$:

$$a\mathbf{e}_1 + b\mathbf{e}_2 = \mathbf{x}\,. \tag{1}$$

The defining feature of a basis such as $\{\mathbf{e}_1, \mathbf{e}_2\}$, and perhaps also its shortcoming, is that the representation given by Equation (1) is *unique*. This means no other constants in addition to a and b would make Equation (1) true.

A *frame*, roughly speaking, emulates this behaviour, but is not restricted by the uniqueness property. This means that frames are nothing more than a non-minimal spanning sets for $\mathbb{R}^2$.

The reconstruction formula (see Equation (7) below) implements a redundant representation of vectors via frames, and it necessarily calls for the computation of the inverse of the so-called *frame operator* associated to the frame. This computation is generally taxing; however, for a certain subset of frames known as *tight frames*, the process simplifies significantly, making their computation trivial (see Proposition 3).

This article aims to provide an exposition on tight frames in $\mathbb{R}^2$, along with their characterisation, as it appears in the textbook [2], which will be our main reference. We shall supply all the computations for the characterisation result of Theorem 4 which were not explicitly worked out in the text. Using the same techniques, we shall also obtain an extension of their results on the frame constant of a tight frame.

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2 Prerequisites

For this article, we mainly restrict ourselves to the 2-dimensional real vector space $\mathbb{R}^2$, more commonly known as the *Cartesian plane* with "x and y coordinate axes". In $\mathbb{R}^2$, any vector $\mathbf{x}$ can be written as a 2-tuple as below:

$$\mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix} . \tag{2}$$

Equation (2) is usually referred to as the Cartesian form of $\mathbf{x}$. On the other hand, we also have the following *polar form* of $\mathbf{x}$:

$$\mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} r\cos(\theta) \\ r\sin(\theta) \end{bmatrix} \tag{3}$$

where r is the *magnitude* of $\mathbf{x}$ and satisfies $r^2 = a^2 + b^2$, and where θ is the measure of the smallest positive angle that the ray connecting $(0,0)$ and (a,b) makes with the positive x-axis. Note that our choice of θ necessarily restricts its value to $[0, 2\pi)$. There may be other ways of defining θ, and in fact another popular way to do it through the so-called "$\mathrm{atan2}$" function.

For the rest of the article, for any vector $\mathbf{x} \in \mathbb{R}^2$, say represented again via Equation (2) with polar form given by Equation (3), we are interested in an associated vector which we dub $\widetilde{\mathbf{x}}$. The vector $\widetilde{\mathbf{x}}$ can be obtained by squaring the magnitude of x and doubling its angle. Explicitly, we have:

$$\widetilde{\mathbf{x}} = \begin{bmatrix} r^2\cos(2\theta) \\ r^2\sin(2\theta) \end{bmatrix} . \tag{4}$$

We shall treat the vector space $\mathbb{R}^2$ also as an *inner product space*, meaning it comes equipped with the *inner product* $\langle \cdot, \cdot \rangle$ which takes two vectors, say $\mathbf{u} = [u_1\ u_2]^{\mathbf{T}}$ and $\mathbf{v} = [v_1\ v_2]^{\mathbf{T}}$, and returns a real number:

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + u_2 v_2 .$$

The inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^2$ induces the so-called *norm* $\| \cdot \|$ on $\mathbb{R}^2$ given by

$$\|\mathbf{u}\| := \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} = \sqrt{u_1^2 + u_2^2} .$$

Hence, one notices that the norm of $\mathbf{u}$ is simply the magnitude of $\mathbf{u}$.

We are now ready to make a most important definition.

Definition 1. Suppose that $k \geq 2$. A collection of vectors $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\} \subseteq \mathbb{R}^2$ is a *frame* if there exist two positive constants $A, B > 0$ such that, for all $\mathbf{x} \in \mathbb{R}^2$,

$$A\|\mathbf{x}\|^2 \leq \sum_{n=1}^{k} \left| \langle \mathbf{x}, \mathbf{x}_n \rangle \right|^2 \leq B\|\mathbf{x}\|^2 . \tag{5}$$

The constants A and B are called the *lower* and *upper frame bounds*, respectively. The collection $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ is said to be a *tight frame* if the lower and upper frame bounds coincide - that is, if $A = B$ - in which case, we call $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ an A-*tight frame*.

For the rest of this section, we shall be interested in some of the basic properties of a frame, all of which can be found in the excellent texts [1, 3], although they are mostly geared towards the theory of frames in infinite-dimension. Our main reference text [2] also contains these results, aimed for those only interested in the finite-dimensional vector space case.

For each collection of elements $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\} \subseteq \mathbb{R}^k$ (which do not necessarily form a frame), there is an associated operator called the *frame operator* S, defined by

$$S\mathbf{x} = \sum_{n=1}^{k} \langle \mathbf{x}, \mathbf{x}_n \rangle \mathbf{x}_n \, . \tag{6}$$

Note that as an operator, S sends vectors in $\mathbb{R}^2$ to $\mathbb{R}^2$, that is, $S : \mathbb{R}^2 \to \mathbb{R}^2$. The frame operator can be used to characterize frames, and in fact the next result motivates why they are interesting mathematical objects.

Theorem 2. *For $k \geq 2$, let $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\} \subseteq \mathbb{R}^2$. Then the following statements are equivalent:*

(a) $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ is a frame;

(b) $S : \mathbb{R}^2 \to \mathbb{R}^2$ is invertible.

If these statements are true, then the canonical reconstruction formula *holds for each $\mathbf{x} \in \mathbb{R}^2$:*

$$\mathbf{x} = SS^{-1}\mathbf{x} = \sum_{n=1}^{k} \langle S^{-1}\mathbf{x}, \mathbf{x}_n \rangle \mathbf{x}_n \, . \tag{7}$$

We see now that frames can be seen as a generalized *basis* for $\mathbb{R}^2$, since any vector $\mathbf{x} \in \mathbb{R}^2$ can be represented by a linear combination of frame elements $\mathbf{x}_1, \ldots, \mathbf{x}_k$ (compare Equation (7) with Equation (1)). For our purposes, we see that the main difference between a frame and a basis is that the total number of frame elements is allowed to exceed the dimension of the vector space. This shows that frames in general are an overcomplete spanning set for $\mathbb{R}^2$. Furthermore, Equation (7) is not unique in any way; that is, for any $\mathbf{x} \in \mathbb{R}^2$, there may be other ways of reconstructing $\mathbf{x}$ by linear combination of $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ that differs from the canonical formula (7). As mentioned in the introduction, this non-uniqueness of the representations finds some use in some applications, particularly in signal processing.

In general, it is not easy to compute the inverse operator S^{-1} whenever one has a frame, and so the reconstruction formula is just as hard to implement as the computation of S^{-1}. However, a certain subclass of frames do not have this problem – enter *tight frames* (see Definition 1).

Proposition 3. *Let $k \geq 2$. If $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ is an A-tight frame for some $A > 0$, that is, if*

$$\sum_{n=1}^{k} |\langle \mathbf{x}, \mathbf{x}_n \rangle|^2 = A\|\mathbf{x}\|^2 \tag{8}$$

for all $x \in \mathbb{R}^2$, then its associated frame operator satisfies

$$S = AI\,, \tag{9}$$

where I is the identity operator on $\mathbb{R}^2$.

As we can see above, A-tight frames have a very simple frame operator, and an equally simple inverse given by Equation (9). This paper aims to show and prove a simple characterisation result for tight-frames in $\mathbb{R}^2$ which also appears in the textbook *Frames for Undergraduates* [2, Lemma 4.1]. The textbook did not provide detailed computations for their corresponding result. The paper aims to fill the gap by providing a full proof and computation, involving only basic trigonometry and matrix algebra – something which is accessible for a motivated high school student. Additionally, the computations in this paper will highlight a simple way to compute the frame bounds for a tight frame.

3 Characterisation of tight frames in $\mathbb{R}^2$

Suppose that one would like to verify whether a vector collection $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\} \subseteq \mathbb{R}^2$ is a tight frame. A straightforward method would be to see if it satisfies Equality (8). However, a different method and geometrically insightful way is also possible. Essentially, we will do this by looking at the correspondence $\mathbf{x} \mapsto \widetilde{\mathbf{x}}$, where $\widetilde{\mathbf{x}}$ is given by the transformation in Equation (4). The following theorem is due to [2, Lemma 4.1].

Theorem 4. *For $k \geq 2$, a collection of vectors $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ in $\mathbb{R}^2$ is a tight frame for $\mathbb{R}^2$ if and only if $\widetilde{\mathbf{x}}_1 + \cdots + \widetilde{\mathbf{x}}_k = \mathbf{0}$.*

Geometrically speaking, if, starting from the origin, the tip-to-tail addition of the transformed vectors $\widetilde{\mathbf{x}}_1, \ldots, \widetilde{\mathbf{x}}_k$ returns to the origin, then the original vectors $\mathbf{x}_1, \ldots, \mathbf{x}_k$ form a tight frame. Below, we have illustrated an example of this.

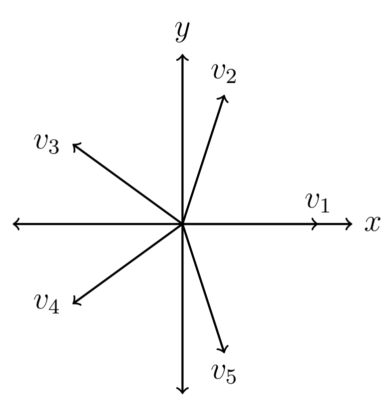

Figure 1: The vectors $\mathbf{v}_1, \ldots, \mathbf{v}_5$ in their original state.

It follows from Figure 3 above that $\mathbf{v}_1, \ldots, \mathbf{v}_5$ is a tight frame in $\mathbb{R}^2$. Curiously, there is no mention of the tight-frame constant A in the statement of the result in Theorem 4.

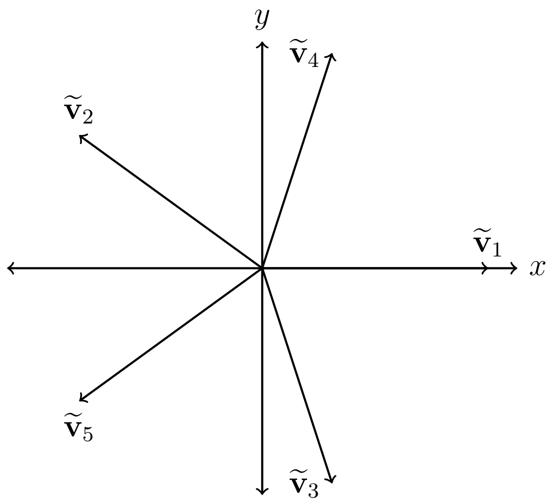

Figure 2: The transformations applied to the vectors.

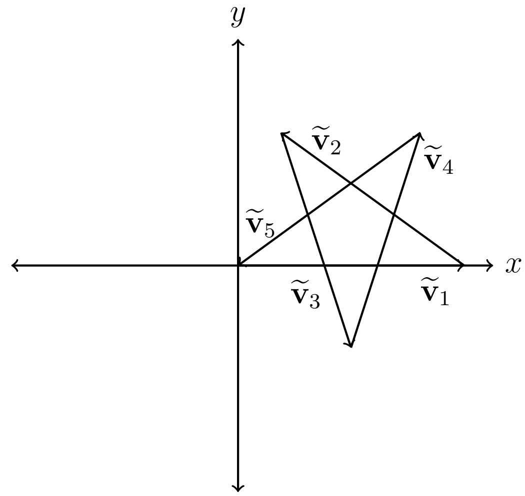

Figure 3: The tip-to-tail vector sum $\widetilde{\mathbf{v}}_1 + \cdots + \widetilde{\mathbf{v}}_5$.

We shall address this discrepancy in Proposition 5, allowing us to explicitly compute the tight frame constant only in terms of the vector elements themselves.

3.1 Initial computations

We are interested in the matrix representation of the frame operator S associated with the elements $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$. That is, we would like to compute the 2×2 matrix $[S_{ij}]$ such that $[S_{ij}]\mathbf{x} = S\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^2$ with $\mathbf{x}$ represented as a column vector. We can use the following formula for each $i, j \in \{1, 2\}$:

$$S_{ij} = \sum_{n=1}^{k} \langle \mathbf{e}_i, \mathbf{x}_n \rangle \langle \mathbf{x}_n, \mathbf{e}_j \rangle \,, \tag{10}$$

where $\mathbf{e}_1 = [1 \;\; 0]^{\mathbf{T}}$ and $\mathbf{e}_2 = [0 \;\; 1]^{\mathbf{T}}$ are the standard basis vectors for $\mathbb{R}^2$.

For the rest of the computations, we shall use the polar representation of each $\mathbf{x}_n$:

$$\mathbf{x}_n = \begin{bmatrix} r_n \cos(\theta_n) \\ r_n \sin(\theta_n) \end{bmatrix} .$$

We first fix j and consider the matrix entries when i is either 1 or 2:

$$\begin{aligned} S_{1j} = \sum_{n=1}^{k} \langle \mathbf{e}_1, \mathbf{x}_n \rangle \langle \mathbf{x}_n, \mathbf{e}_j \rangle &= \sum_{n=1}^{k} \left\langle \mathbf{e}_1, \begin{bmatrix} r_n \cos(\theta_n) \\ r_n \sin(\theta_n) \end{bmatrix} \right\rangle \left\langle \begin{bmatrix} r_n \cos(\theta_n) \\ r_n \sin(\theta_n) \end{bmatrix}, \mathbf{e}_j \right\rangle \\ &= \sum_{n=1}^{k} r_n \cos(\theta_n) \left\langle \begin{bmatrix} r_n \cos(\theta_n) \\ r_n \sin(\theta_n) \end{bmatrix}, \mathbf{e}_j \right\rangle \end{aligned}$$

$$S_{2j} = \sum_{n=1}^{k} \langle \mathbf{e}_1, \mathbf{x}_n \rangle \langle \mathbf{x}_n, \mathbf{e}_j \rangle = \sum_{n=1}^{k} \left\langle \mathbf{e}_2, \begin{bmatrix} r_n \cos(\theta_n) \\ r_n \sin(\theta_n) \end{bmatrix} \right\rangle \left\langle \begin{bmatrix} r_n \cos(\theta_n) \\ r_n \sin(\theta_n) \end{bmatrix}, \mathbf{e}_j \right\rangle$$
$$= \sum_{n=1}^{k} r_n \sin(\theta_n) \left\langle \begin{bmatrix} r_n \cos(\theta_n) \\ r_n \sin(\theta_n) \end{bmatrix}, \mathbf{e}_j \right\rangle .$$

Simple substitutions on j now yield the following matrix representation of the frame operator S associated with $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$:

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \begin{bmatrix} \sum_{n=1}^{k} r_n^2 \cos^2(\theta_n) & \sum_{n=1}^{k} r_n^2 \cos(\theta_n) \sin(\theta_n) \\ \sum_{n=1}^{k} r_n^2 \sin(\theta_n) \cos(\theta_n) & \sum_{n=1}^{k} r_n^2 \sin^2(\theta_n) \end{bmatrix} . \tag{11}$$

3.2 Proof of Proposition 4

To prove Proposition 4, we use Equation (9) of Proposition 3 which says that $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ is a tight frame if and only if $S = AI$ for some constant $A > 0$. It follows from (11) that

$$\begin{bmatrix} \sum_{n=1}^{k} r_n^2 \cos^2(\theta_n) & \sum_{n=1}^{k} r_n^2 \cos(\theta_n) \sin(\theta_n) \\ \sum_{n=1}^{k} r_n^2 \sin(\theta_n) \cos(\theta_n) & \sum_{n=1}^{k} r_n^2 \sin^2(\theta_n) \end{bmatrix} = S = AI = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} . \tag{12}$$

Therefore, $S_{11} - S_{22} = A - A = 0$, so

$$\sum_{n=1}^{k} r_n^2 (\cos^2(\theta_n) - \sin^2(\theta_n)) = \sum_{n=1}^{k} r_n^2 \cos(2\theta_n) = 0 , \tag{13}$$

where we used the double-angle formula $\cos^2(\theta) - \sin^2(\theta) = \cos(2\theta)$. On the other hand, $S_{12} + S_{21} = 0 + 0 = 0$, so it follows from $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ that

$$\sum_{n=1}^{k} r_n^2 (2\sin(\theta_n)\cos(\theta_n)) = \sum_{n=1}^{k} r_n^2 \sin(2\theta_n) = 0 . \tag{14}$$

To reiterate, we have from Equations (13) and (14) that

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \sum_{n=1}^{k} r_n^2 \sin(2\theta_n) \\ \sum_{n=1}^{k} r_n^2 \cos(2\theta_n) \end{bmatrix} = \sum_{n=1}^{k} \begin{bmatrix} r_n^2 \sin(2\theta_n) \\ r_n^2 \cos(2\theta_n) \end{bmatrix} = \sum_{n=1}^{k} \widetilde{\mathbf{x}}_n . \tag{15}$$

We have shown that if $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ is a tight frame, then $\sum_{n=1}^{k} \widetilde{\mathbf{x}}_n = \mathbf{0}$.

The converse essentially works by just reversing our computations, but we shall still detail it here for completeness.

Suppose that $\sum_{n=1}^{k} \widetilde{\mathbf{x}}_n = \mathbf{0}$ and let $A := \sum_{n=1}^{k} r_n^2 \cos(\theta_n)$.

If $A = 0$, then, for all n, either $r_n = 0$ or $\theta_n = \pi m_n/2$ for some $m_n \in \mathbb{Z}$. This cannot happen because then all $\mathbf{x}_n$ would be scalar multiples of the standard basis $\mathbf{e}_2$ and could therefore not span $\mathbb{R}^2$. Therefore, $A > 0$. It follows from our choice of A and Equation (11) that $S_{11} - S_{22} = A - S_{22}$, but then the hypothesis $\sum_{n=1}^{k} \widetilde{\mathbf{x}}_n = \mathbf{0}$ implies that $S_{11} - S_{22} = 0$ (see Equation (14)). Therefore, $S_{11} = S_{22} = A$. On the other hand, the hypothesis $\sum_{n=1}^{k} \widetilde{\mathbf{x}}_n = \mathbf{0}$ also implies that $S_{21} + S_{12} = 0$. However, we know from Equation (11) that $S_{12} = S_{21} \geq 0$. Therefore, it must be that $S_{12} = S_{21} = 0$. We have now shown that there exists an $A > 0$ such that $S = AI$; therefore S is an A-tight frame. □

3.3 Computing the Frame Constant

Observe that the proof above only utilized the difference between the diagonal entries in (12). We can also use the sum of diagonal entries to obtain a result concerning the tight frame constant A.

Proposition 5. *For $k \geq 2$, let $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ be a set of vectors in $\mathbb{R}^2$ with polar representations*

$$\mathbf{x}_n = \begin{bmatrix} r_n \cos(\theta_n) \\ r_n \sin(\theta_n) \end{bmatrix}.$$

If $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ is a tight-frame for $\mathbb{R}^2$ with frame constant A, then

$$A = \sum_{n=1}^{k} \frac{r_n^2}{2}.$$

Proof. Suppose that $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ is a tight-frame with frame constant A. Then the associated frame operator S satisfies $S = AI$. It follows again from Equation (12) that

$$2A = S_{11} + S_{22} = \sum_{n=1}^{k} r_n^2 \cos^2(\theta_n) + \sum_{n=1}^{k} r_n^2 \sin^2(\theta_n) = \sum_{n=1}^{k} r_n^2(\cos^2(\theta_n) + \sin^2(\theta_n)) = \sum_{n=1}^{k} r_n^2.$$

The proposition follows. □

4 Conclusion

The computations above show that, by using the transformation $\mathbf{x} \mapsto \widetilde{\mathbf{x}}$, a set of vectors in $\mathbb{R}^2$ can be verified to be a tight frame by simple trigonometry and algebra. Furthermore, we were able to give an explicit way of computing the tight frame constant.

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