There is only one theorem in A-level calculus

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1   Introduction

1.1   Background

Reverse mathematics [1] is a field devoted to examining which theorems – typically expressed in logic – can be considered to be equivalent, given a presupposed set of assumptions (known as a base theory and, again, typically expressed in logic). Alternatively, reverse mathematics can be viewed as the search for the minimum axioms required to prove theorems; as if – presupposing and excepting the base theory – a theorem implies all the axioms that have already been used to prove it, then those axioms must be *required* to prove the theorem, and no subset of those axioms will suffice to prove the theorem.

This notion of "minimum axioms" is entirely equivalent to the notion of equivalent theorems. If a collection of theorems and axioms is the minimum required axioms, aside from the base theory, to prove another theorem, then this collection of theorems is equivalent to the second theorem. These are dual approaches and both are valid but herein we adopt the perspective of considering which theorems are equivalent over the base theory.

For simplicity purposes, throughout this article I will not consider theorems and axioms in terms of logic. A single strategy will be used to establish the equivalence of theorems: if we are given two theorems A and B, they are shown to be equivalent over the base theory by both

  (a) presupposing Theorem A along with the base theory to derive Theorem B, and

  (b) presupposing Theorem B along with the base theory to derive Theorem A.

The proposed base theory is very simple. In fact, the point of the base theory is to be minimal: in an ideal world, the base theory would be completely empty; i.e. there would be no base theory, but this is impractical, not impossible, as some basic principles, axioms, and assumptions are needed to reason with and derive theorems from other theorems.

If an attempt were made to prove two theorems were equivalent without a base theory, then there would be no reasoning that could get from one theorem to the other. They would each be inert and isolated, with nothing to interact with to imply the other.

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Keeping this in mind, the base theory in this article is constructed to assume as little as possible, while still keeping basic reasoning possible. The assumptions in this base theory are essentially the most basic facts of mathematics and a minimal amount of calculus. It reflects just slightly more than a GCSE student would know about calculus and the foundations of mathematics when first progressing to A-level mathematics. I refer to this base theory as GCSE^+.

For non-British readers, I will pause for a moment to clarify what I mean by a GCSE student. GCSEs are the first public examinations British students sit (although in Scotland they use an alternative) at age 16. Calculus is typically first introduced to students almost immediately after GCSEs when or if they progress to the next qualification: A-levels.

1.2 Basic calculus notation and definitions

The basic notation and definitions used in this article, which many readers should already be familiar with, are compiled below:

(1) The **derivative** of a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is denoted by f'. No definition of the derivative in terms of limits or Newton quotients is given, as this then assumes much more. Additionally, for any natural number k, $f^{(k)}(x)$ denotes the kth derivative of a function f with respect to x.

(2) An **indefinite integral** of a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is denoted by $\int f(x)\,dx$. Note that the indefinite integral is not *yet* defined to be the inverse of differentiation: this will be assumed in the base theory later.

(3) The **definite integral** of a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ between the limits $a, b \in \mathbb{R}$, where $a \leq b$, is denoted by $\int_a^b f(x)\,dx$ and has the value $F(b) - F(a)$, where F is an indefinite integral of f.

(4) Impose an ordering on $\mathbb{R}$ as follows: for all $x, y \in \mathbb{R}$,

    (a) $x > y$ if and only if $x - y$ is positive;

    (b) $x \geq y$ if and only if $x = y$ or $x - y$ is positive;

    (c) $x < y$ if $x \geq y$ is not true;

    (d) $x \leq y$ if $x > y$ is not true.

(5) The absolute value of any $y \in \mathbb{R}$ is defined to be

$$|y| = \begin{cases} y & \text{if } y \text{ is positive or zero;} \\ -y & \text{if } y \text{ is negative.} \end{cases}$$

1.3 Base theory

On top of the arithmetic properties of the real numbers, the law of trichotomy of real numbers that any real number is either positive, negative or zero, and the standard foundational tools of deductive reasoning (e.g., mathematical induction), the only additional assumptions are:

(a) All functions considered here are differentiable and integrable, and remain so as many times as is needed. Their domains and ranges are always $\mathbb{R}$.

(b) For each $n \in \mathbb{N}$, $\left(x^n\right)' = nx^{n-1}$ and the derivative of any constant is 0.

(c) For any function f, the integral and derivative are inverses of each other; that is,

$$\left(\int f(x)dx\right)' = \int \left(f(x)\right)'dx = f(x).$$

(d) Differentiation is linear, in the sense that $(\alpha u + \beta v)' = \alpha u' + \beta v'$ for any function u, v and $\alpha, \beta \in \mathbb{R}$.

(e) The integral of a function that is always positive (respectively zero) is positive (respectively zero).

(f) For any infinite series such that the sum of the first $n + 1$ terms always better approximates a function than the sum of the first n terms on some specified domain, the sum of the entire infinite series equals that function on that domain.

(g) For any function f considered herein and interval $[a, b] \subset \mathbb{R}$, there exists some constant $C \in \mathbb{R}$ such that for each $k \in \mathbb{R}$ and $x \in [a, b]$,

$$\left|f^{(k)}(x)\right| < C^{k+1}k!.$$

This is equivalent to assuming all functions considered here are analytic.

1.4 Statements of theorems considered

The theorems to be proved equivalent, to reflect that the base theory is approximately GCSE-level mathematics, are chosen to be hallmarks of A-level calculus: the product rule, Taylor's theorem, and integration by parts. These are stated below.

Theorem 1 (The product rule [2]). *For any function u and v, we have that*

$$(uv)' = u'v + uv'. \tag{1}$$

Theorem 2 (Integration by parts [3]). *For any function u and v, we have that*

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx. \tag{2}$$

Theorem 3 (Taylor's theorem [4]). *Let $k \in \mathbb{N}$ and $p \in \mathbb{R}$ be arbitrary. For any function f, we have that*

$$f(x) = \sum_{j=0}^{k} \frac{f^{(j)}(p)}{j!}(x-p)^j + \int_p^x \frac{f^{(k+1)}(y)}{k!}(x-y)^k \, dy \,. \tag{3}$$

Note 1. *For clarity, herein theorems about theorems and what axiom systems they can be proved in are labeled as Meta-Theorems.*

2 Preliminary theorems

Note that there are already some theorems that can be proven within GCSE^+ and will be useful later. For example, Theorem 4; which is shown to be be provable using only GCSE^+ in Meta-theorem 1.

Theorem 4. *For any $n \in \mathbb{N}$, $\int x^n \, dx = \dfrac{x^{n+1}}{n+1} + c$ for some $c \in \mathbb{R}$.*

Meta-theorem 1. *Theorem 4 may be proven within GCSE^+.*

Also note that there are lemmas that can be proven to be implied by any of the three above theorems but do not necessarily imply them back. This point is illustrated with Meta-lemma 1 but, before presenting and proving it, there is a pause to define an important concept required by it.

Definition 5. *A **neighbourhood** of a specified point is an area (e.g., an interval if in one dimension) around that point of unspecified size (and shape if in more than one dimension).*

Meta-lemma 1. *Assuming GCSE^+ and Taylor's Theorem, for any function f, there exists some neighbourhood of any point $p \in \mathbb{R}$, such that for any $x \in \mathbb{R}$ within that neighbourhood, we have that*

$$f(x) = \sum_{j=0}^{\infty} \frac{f^{(j)}(p)}{j!}(x-p)^j \,.$$

Proof. From assumption (g) of the base theory, within any interval centered on $p \in \mathbb{R}$, there exists some constant $C \in \mathbb{R}$ such that $\left| f^{(k)}(x) \right| < C^{k+1} k!$. Hence,

$$\frac{\left| f^{(k)}(x) \right|}{k!}|x-y|^k < \frac{C^{k+1} k!}{k!}|x-y|^k = C^{k+1}|x-y|^k \,.$$

This is then used to show that for all $k \in \mathbb{N}$,

$$\int_p^x \frac{f^{(k+1)}(x)}{k!}(x-y)^k \, dy \leq \left| \int_p^x \frac{f^{(k+1)}(y)}{k!}(x-y)^k \, dy \right|$$

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$$\leq \int_p^x \frac{\left|f^{(k+1)}(y)\right|}{k!} |x-y|^k \, dy \leq C'^{k+1} \int_p^x |x-y|^k \, dy . \qquad (4)$$

If x is restricted to be within ξ^{-1} of p, where $\xi = 1.1C$, then if y is restricted to within the limits of the integral in (4), it follows that $|x-y| \leq \xi^{-1}$ and $x - p \leq \xi^{-1}$. Therefore, using this restriction,

$$\int_p^x \frac{f^{(k+1)}(x)}{k!}(x-y)^k dy \leq C'^{k+1} \int_p^x |x-y|^k dy \leq C^{k+1}\xi^{-k}(x-p) \leq (1.1)^{-k} . \qquad (5)$$

It can be shown that, within GCSE$^+$, $(1.1)^{-k} > (1.1)^{-(k+1)}$ for all $k \in \mathbb{N}$, so the series can be extended to infinitely many terms with zero error by assumption (f). In other words,

$$f(x) = \sum_{j=0}^{\infty} \frac{f^{(j)}(p)}{j!}(x-p)^j$$

and the proof is complete. □

3   Results

The main result of this paper is Meta-theorem 2, which is proven in the subsequent subsections.

Meta-theorem 2. *Taylor's theorem, the product rule, and integration by parts are equivalent over GCSE$^+$.*

This theorem will be proven via a sequence of disjoint proofs. These will collectively show that

the product rule $\Rightarrow$ integration by parts $\Rightarrow$ Taylor's theorem $\Rightarrow$ the product rule,

proving the equivalence of the three theorems over $GCSE^+$.

3.1   Product rule implies integration by parts

Let u and v be differentiable functions of x and start from the product rule in Definition 1:

$$(uv)' = u'v + uv' . \qquad (6)$$

By taking the integral of both sides of (6) and using the linearity of integration (which can be proven within GCSE$^+$), we have that

$$\int (uv)' \, dx = \int (u'v + uv') \, dx = \int u'v \, dx + \int uv' \, dx . \qquad (7)$$

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Hence, applying the Fundamental Theorem of Calculus to the leftmost part of (7) gives

$$uv = \int u'v\,dx + \int uv'\,dx$$

which can then be rearranged to give the following formula for integration by parts:

$$\int uv'\,dx = uv - \int u'v\,dx\,.$$

Note 2. *This argument can easily be reversed to show that integration by parts implies the product rule.*

3.2 Integration by parts implies Taylor's theorem

Take an arbitrary function g and $p \in \mathbb{R}$. Define a new function f by $f(x) = g(x - p)$. We start by defining

$$J[x, n] = \int_0^x \frac{f^{(n+1)}(y)}{n!}(x - y)^n dy\,. \tag{8}$$

By integrating the integral in (8) by parts, we get

$$J[x, n] = \left[\frac{f^{(n+1)}(y)}{n!}\int (x - y)^n dy\right]_0^x - \int_0^x \left(\frac{f^{(n+2)}(y)}{n!}\int (x - y)^n\,dy\right) dy. \tag{9}$$

The integral $\int (x - y)^n\,dy$ in (9) can be evaluated given GCSE^+ and integration by parts (using that it implies the product rule, as discussed above). Hence

$$J[x, n] = \frac{f^{(n+1)}(y)}{(n + 1)!}(x - y)^{n+1}\bigg|_{y=x}^{y=0} + \int_0^x \frac{f^{(n+2)}(y)}{(n + 1)!}(x - y)^{n+1}\,dy$$

$$= \frac{f^{(n+1)}(0)}{(n + 1)!}x^{n+1} + J[x, n + 1].$$

We prove the existence of a Taylor series of f about $x = 0$ for any value of n by induction. By the Fundamental Theorem of Calculus,

$$f(x) - f(0) = \int_0^x f'(y)\,dy$$

which implies that

$$f(x) = f(0) + \int_0^x f'(y)\,dy = f(0) + J[x, 0].$$

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This is the required Taylor series of f about $x = 0$ when $n = 0$, establishing the base case. Assume that there exists a Taylor series expansion of f about $x = 0$ for a given value of $n \in \mathbb{N}$. Then

$$f(x) = \sum_{j=0}^{n} \frac{f^{(j)}(0)}{j!} x^j + \int_0^x \frac{f^{(n+1)}(y)}{n!} (x - y)^n \, dy = \sum_{j=0}^{n} \frac{f^{(j)}(0)}{j!} x^j + J[x, n] \,.$$

Then, using (9), it follows that

$$\begin{aligned} f(x) &= \sum_{j=0}^{n} \frac{f^{(j)}(0)}{j!} x^j + \frac{f^{(n+1)}(0)}{(n+1)!} x^{n+1} + J[x, n+1] \\ &= \sum_{j=0}^{n+1} \frac{f^{(j)}(0)}{j!} x^j + J[x, n+1] \,. \end{aligned}$$

Therefore, the required Taylor series of f about $x = 0$ with $n + 1$ terms exists and by induction, f has a Taylor series expansion with any positive number of terms.

To complete the proof, note that $g^{(n)}(p) = f^{(n)}(0)$ for any non-negative integer n, therefore proving that g has a Taylor series expansion about $x = p$ with any positive number of terms.

3.2.1 Taylor's theorem implies the product rule

Let u and v be functions of x. By assumption and Lemma 1, there exist some neighbourhood $I_u^{(s)}, I_v^{(s)} \subseteq \mathbb{R}$ of any point $s \in \mathbb{R}$ such that

$$u(x) = u_0 + u_1(x - s) + u_2(x - s)^2 + \cdots$$

for all $x \in I_u^{(s)}$ and

$$v(x) = v_0 + v_1(x - s) + v_2(x - s)^2 + \cdots$$

for all $x \in I_v^{(s)}$. Take any $x \in I_u^{(s)} \cap I_v^{(s)}$. Then we have that

$$\big(u(x)v(x)\big)' = \left(\sum_{k=0}^{\infty} \sum_{\substack{i,j: \\ i+j=k}} u_i v_j (x - s)^k \right)' = \sum_{k=0}^{\infty} k \sum_{\substack{i,j: \\ i+j=k}} u_i v_j (x - s)^{k-1} \,. \tag{10}$$

On the other hand,

$$\begin{aligned} u'(x)v(x) &= \left(\sum_{j=0}^{\infty} u_j (x - s)^j \right)' \big(v_0 + v_1(x - s) + \cdots\big) \\ &= \left(\sum_{j=0}^{\infty} j u_j (x - s)^{j-1} \right) \big(v_0 + v_1(x - s) + \cdots\big) \end{aligned}$$

$$= \sum_{k=0}^{\infty} \sum_{\substack{i,j:\\i+j=k}} ju_j v_i (x-s)^k.$$

Using a similar argument to derive $u(x)v'(x)$, it can be calculated that

$$u'(x)v(x) + u(x)v'(x) = \sum_{k=0}^{\infty} \sum_{\substack{i,j:\\i+j=k}} (i+j)u_j v_i (x-s)^k$$

$$= \sum_{k=0}^{\infty} \sum_{\substack{i,j:\\i+j=k}} ku_j v_i (x-s)^k$$

$$= \big(u(x)v(x)\big)'$$

for all $x \in I_u^{(s)} \cap I_v^{(s)}$, where the last equality follows from (10).

This – *prima facie* – appears to be the product rule, as required. However, thus far, the product rule has only been shown to hold for x within $I_u^{(s)} \cap I_v^{(s)}$. Since s was arbitrary, for any $x_0 \in \mathbb{R}$ there always exists a $s_0 \in \mathbb{R}$ such that $x_0 \in I_u^{(s_0)} \cap I_v^{(s_0)}$. Hence, the above result can be generalized to hold for any $x_0 \in \mathbb{R}$.

4   Discussion

Herein, a base theory which is called GCSE^+ – roughly equivalent to GCSE mathematics supplemented with very basic calculus – has been presented, over which many of the major theorems in A-level mathematics have been shown to be equivalent. This is interesting for many reasons, among them is that – given the cumulative nature of mathematics – and that the line where GCSE mathematics ends and A-level mathematics begins is somewhat arbitrary – from a strictly mathematical perspective. This makes it pertinent to ask: at what other points in the progressive accumulation of mathematical theorems are the next few theorems equivalent, using all accumulated mathematics to that point as the base theory. It is a topic for future research to investigate the density of these points but given the base theory was not heavily engineered to make the equivalence possible and that using what is essentially GCSE mathematics as a base theory – which in a way serves as a "nothing-up-my-sleeve" base theory – these points appear extremely common.

Another topic for future research – that the present author particularly encourages readers newer to research mathematics to take up – is extending the list of theorems that are equivalent over GCSE^+. It did not take much searching to find the theorems shown to be equivalent, over GCSE^+, herein, which suggests that much "low-hanging fruit" is still available to show is equivalent over GCSE^+. It also suggests that theorems equivalent to those presented in earlier sections are very common. This is a phenomenon observed in reverse mathematics, where there are only five main classes of theorems that are equivalent, despite hundreds of theorems having been considered.

5 Acknowledgements

References

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